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# Non-Linear Force Dynamics in Black Hole Accretion Disks Using Logarithmic and Trigonometric Approximations

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**Abstract:** Accretion disks are material structures that form when gas, dust or plasma is pulled towards a black hole by its strong gravity. In rotating (Kerr) black holes, the dynamics of the accretion disk becomes more complex due to frame-dragging effects that affect particle trajectories and energy release patterns. This study aims to develop an algorithm for calculating the force F(x) on particles in the accretion disk by considering the non-linear interaction between the black hole spin parameter and the relative position of the particles. The developed algorithm uses logarithmic and trigonometric approaches to improve the accuracy of the force calculation. The results show that variations in the spin parameter and relative position significantly affect the force distribution in the accretion disk. Visualization of the force interaction reveals the existence of non-linear patterns that contribute to the system dynamics. The main contribution of this research is the refinement of force calculation models that previously did not fully incorporate the combined effects of logarithmic and trigonometry in particle interactions. The proposed approach offers a more accurate predictive tool to explore the physical processes around black holes, and supports the interpretation of observational data from telescopes and gravitational wave detectors.

**Keywords:** Accretion disks; Force calculations; Non-linear dynamics; Rotating black holes; Spin parameters.

## Introduction

An accretion disk is a structure of material that forms when gas, dust or plasma is pulled toward a black hole by its intense gravity, but does not fall directly into it (Malki, 2024; Pugliese & Stuchlík, 2021). This matter forms a disk-shaped structure that rotates at high speeds due to angular momentum. Close to the black hole, radiation pressure, the viscosity of the matter in the disk, and the effects of general relativity become more dominant (Pugliese & Stuchlík, 2021; Wessel et al., 2021). In rotating black holes, known as Kerr black holes, the dynamics of the accretion disk become more complex (Ashoorioon et al., 2024; Hou et al., 2022). The rotation of the black hole creates a gravitational effect called frame dragging, where space-time around the black hole is dragged along by the rotation (Kraniotis, 2021; Maes, 2023). This effect affects the orbit of the material in the accretion disk and the pattern of energy released. As a result, accretion disks are among the brightest sources of electromagnetic radiation in the Universe, especially in the form of X-rays and gamma rays (Anitra, 2024; Boruah et al., 2024). This radiation is produced when the material in the disk experiences extreme heating due to friction and compression as it approaches the black hole (Bourne et al., 2024; Turimov et al., 2024). Supermassive black holes at the center of galaxies often have active accretion disks, and the interaction between the radiation produced by these disks and the surrounding gas can affect star formation and the distribution of mass in the galaxy (Schinnerer & Leroy, 2024; Volonteri et al., 2021). Therefore, understanding the dynamics of

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accretion disks is not only important in the study of black holes, but also in galaxy astrophysics in general.

One of the main challenges in understanding accretion disks is how the rotational effects of the black hole affect the distribution of forces acting on particles in the disk. The black hole's spin creates a strong and asymmetric gravitational field, which affects the distribution and behavior of particles in the accretion disk (Cattorini & Giacomazzo, 2024; Reynolds, 2021). Gravitational interactions between particles cause significant turbulence, which affects the pattern of matter flow in the disk (Baehr & Zhu, 2021; Béthune et al., 2021). Particles closer to the black hole experience much stronger tidal forces than those further away (Kostić et al., 2009; Rossi et al., 2021), causing variations in pressure, temperature, and density within the disk. Relativistic effects of black hole spin, such as frame dragging, change the angular momentum of particles in the disk, causing changes in orbital energy and direction of motion (Banerjee et al., 2022; Pugliese & Stuchlík, 2021). These interactions are non-linear, where a small change in one of the parameters can have a significant impact on the dynamics of the entire system. Therefore, modeling the forces in an accretion disk requires a more accurate computational approach to capture the complexity of this system.

This research focuses on the development and implementation of an algorithm to calculate the forces acting on particles in an accretion disk around a rotating black hole. The mathematical model used incorporates logarithmic and trigonometric functions to improve the accuracy of the force calculations, leading to a deeper understanding of accretion disk dynamics and black hole physics in general. The novelty of this research lies in the algorithmic approach that is able to account for the interaction between the spin parameter and the relative position of the particles in the force calculation, something that has not been accounted for in many existing models. Many studies have examined the dynamics of accretion disks, but there is still a gap in computational methods that consider the simultaneous influence of relativistic effects and the internal structure of the disk. Existing models often neglect logarithmic and trigonometric interactions, which are actually very important in characterizing the forces in these systems. By introducing a new algorithm that comprehensively integrates spin parameters with relative positions, this study provides a more accurate solution in modeling the complex interactions in accretion disks. The algorithm provides a systematic framework for calculating the force F(x) over a wide range of spin and radius parameters. This approach not only improves the understanding of accretion disk dynamics, but also fills a gap in the field by providing a more powerful and accurate tool to explore the physical processes around black holes.

This research supports the development of predictive models of black hole behavior, with potential applications in the study of high-energy astrophysical phenomena, such as relativistic jet formation, X-ray emission, and accretion variability. The approach can be used in the interpretation of observational data from telescopes and gravitational wave detectors, which can help in observing and understanding the accretion disk phenomenon in greater depth. However, this study has limitations as it relies on the Kerr metric and the thin-disk approach, which may not fully reflect the complexity of real accretion disks. The algorithm also uses certain idealization conditions, such as the spin parameter range ($a_* \in [-1,1]$) and radius value ($x \geq 1.5$), which may limit its application in some astrophysical scenarios.

This research offers an innovative approach to understanding the forces in an accretion disk through the development of more accurate algorithms. By considering relativistic effects, particle interactions, and non-linear influences in the system, the model can make important contributions to the study of accretion disk dynamics and black hole physics more broadly.

*Kerr Metric and Spacetime Around Rotating Black Holes*

The Kerr metric provides the solution to Einstein's field equations for the spacetime geometry around a rotating black hole (O'Neill, 2014; Teukolsky, 2015). This metric is of significant importance for analyzing the dynamics of particles, particularly those in the accretion disk around the black hole. The Kerr metric is characterized by the mass (M) of the black hole, its angular momentum (J), and its spin parameter (a), where the spin parameter (a) is defined as (Li & Bambi, 2014; Sheoran et al., 2018).

$$a = \frac{J}{M} \quad \text{with} \quad 0 \leq a \leq M \tag{1}$$

This metric can be written in Boyer–Lindquist coordinates $(t, r, \theta, \phi)$, where the components of the metric tensor $(g_{\mu\nu})$ are given by:

$$ds^2 = -\left\{\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 + \frac{4Mar\sin^2\theta}{\rho^2}dtd\phi\right\} + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2 \tag{2}$$

where:

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2 \tag{3}$$

The accretion disk structure around a rotating black hole is influenced by the spacetime curvature, which is governed by the black hole's spin, mass, and the surrounding environment (Pugliese & Stuchlík, 2021). The dynamics of particles within the accretion disk are governed by energy (E), angular momentum ($L_z$), and the spin parameter (a) of the black hole (Ashoorioon et al., 2024). The effective potential governing the motion of test particles in the equatorial plane of the Kerr black hole can be expressed as:

$$V_{\text{eff}}(r) = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L_z^2}{r^2}\right) - \frac{2aML_z}{r} \tag{4}$$

Here, ($L_z$) is the angular momentum of the particle about the black hole, and (a) is the spin parameter. The radial equilibrium condition for the accretion disk can be derived by considering the first derivative of the effective potential $V_{\text{eff}}(r)$ with respect to (r):

$$\frac{dV_{\text{eff}}(r)}{dr} = 0 \tag{5}$$

Solving this equation gives the locations of the stable circular orbits in the accretion disk, which can be further refined based on energy and angular momentum parameters.

The spin of the black hole plays a crucial role in shaping the properties of the accretion disk, particularly in terms of the innermost stable circular orbit (ISCO) and the overall dynamics of the disk (Abramowicz & Fragile, 2013; Reynolds, 2021). The ISCO, which is the closest orbit that a particle can maintain without spiraling into the event horizon, is defined by the following relationship for a rotating black hole:

$$r_{\text{ISCO}} = 3M + \frac{2M}{1 + \cos\theta_{\text{ISCO}}} \tag{6}$$

Where $(\theta_{\text{ISCO}})$ is related to the spin parameter (a) through the trigonometric cosine function. The radius of the ISCO decreases as the spin parameter (a) increases, meaning that faster rotating black holes can accrete matter closer to the event horizon (Middleton, 2016). For an accretion disk, the total bolometric luminosity $(L_{\text{bol}})$ is the sum of the electromagnetic radiation emitted by the disk across all wavelengths (Anitra, 2024). It can be calculated by integrating the luminosity over the emission spectrum:

$$L_{\text{bol}} = \int_{\lambda_{\min}}^{\lambda_{\max}} L_\lambda \, d\lambda \tag{7}$$

Where $(L_\lambda)$ is the luminosity per unit wavelength. In the case of an accretion disk around a rotating black hole, the luminosity is primarily due to the gravitational energy released by the accreting material (Bisnovatyi-Kogan & Lovelace, 2001). As the material spirals inward, it is accelerated by the black hole's gravitational field, emitting radiation as it heats up due to friction and other dissipative processes (Punsly, 2008).

For a particle moving in the accretion disk, the equation of motion can be derived from the Lagrangian in the Kerr metric (Ashoorioon et al., 2024). The effective force F(x) acting on a particle in the disk is governed by a combination of its energy (E), angular momentum ($L_z$), and the spin parameter (a) (Fung & Wong, 2015). This force can be expressed as an advanced equation that includes logarithmic and trigonometric functions, which better describe the particle dynamics in the strong gravitational field of the black hole:

$$F(x) = \frac{GM}{x^2}\left\{\log\left(\frac{x}{x_0}\right) + \frac{\sin\left(\frac{\pi}{2} - \arccos(a)\right)}{x_0}\right\} \tag{8}$$

Here, ($x_0$) is the characteristic radius associated with the ISCO, and (x) is the relative radial position of the particle.

*Total Energy and Angular Momentum in Accretion Disks*

The total energy (E) of a particle in the accretion disk of a rotating black hole is composed of its rest mass energy, kinetic energy, and potential energy, all of which are affected by the spacetime curvature in the Kerr metric (Crinquand, 2021). The total energy can be expressed as:

$$E = -g_{\mu\nu} u^\mu u^\nu, \quad (\mu, \nu = t, r, \theta, \phi) \tag{9}$$

Where $(g_{\mu\nu})$ are the components of the Kerr metric, $(u^\mu)$ is the four-velocity of the particle, The energy expression is derived from the temporal component of the four-momentum (Morales-Herrera et al., 2024). For a particle in an accretion disk, the total energy (E) at a radius (r) in the equatorial plane of the Kerr black hole can be approximated as (Gimeno-Soler et al., 2021):

$$E(r) = \sqrt{-g_{tt} - 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2} \tag{10}$$

Where $(\Omega = d\phi/dt)$ is the angular velocity of the particle, $(g_{tt}, g_{t\phi}, g_{\phi\phi})$ are the components of the Kerr metric in the equatorial plane. In this case, E(r) is influenced by the spin parameter (a) of the black hole and the particle's radial position (r). The angular momentum ($L_z$) of a particle in the accretion disk is defined as:

$$L_z = -g_{\mu\nu} u^{\mu} r^{\nu} \tag{11}$$

For a particle moving in the equatorial plane, this simplifies to:

$$L_z = mr^2\Omega \tag{12}$$

Here, $(\Omega)$ is the angular velocity of the particle, and it depends on both the radial coordinate (r) and the spin parameter (a) of the black hole. The spin parameter modifies the geometry of spacetime, and thus the value of $(\Omega)$. The angular momentum can also be written in terms of the energy (E) and the radial distance (r) using the relativistic energy and angular momentum conservation equations:

$$E^2 = f(r) + \frac{L_z^2}{r^2} \tag{13}$$

This equation governs the motion of particles in the accretion disk, illustrating the relationship between energy and angular momentum. The radial equation of motion for a particle in the accretion disk can be written as:

$$\left(\frac{dr}{dt}\right)^2 = \frac{r^4}{(r^2+a^2)^2}\left\{E^2 - f(r) - \frac{L_z^2}{r^2}\right\} \tag{14}$$

Where f(r) is a potential function that depends on the radius and the black hole's spin parameter, $(dr/dt)$ represents the change in the radial position of the particle with respect to time. This equation describes how the radial position of the particle evolves over time in response to the conservation of energy and angular momentum, with both the black hole spin and the particle's initial conditions determining the outcome (Reynolds, 2021). By combining the energy and angular momentum equations, the relationships governing the motion and distribution of particles in the accretion disk are obtained (Pugliese & Stuchlík, 2021). The relationship between the energy (E), angular momentum ($L_z$), and the radius (r) can be simplified into an effective potential form:

$$V_{\text{eff}}(r) = f(r) + \frac{L_z^2}{r^2} \tag{15}$$

Where the potential $V_{\text{eff}}(r)$ determines the equilibrium positions of particles in the disk. The minima of this potential correspond to the stable orbits where particles can remain in equilibrium, with the innermost stable circular orbit (ISCO) being the closest stable orbit to the black hole (Kagohashi et al., 2024). The stability of these orbits is determined by the interplay between the total energy (E) and the angular momentum ($L_z$) (Siagian et al., 2023). As the spin parameter (a) increases, the ISCO radius decreases, allowing for closer orbits and altering the overall structure of the accretion disk.

To fully describe the dynamics of particles in an accretion disk, we can use the relativistic Euler-Lagrange equations in the Kerr spacetime, which express the conservation of energy and angular momentum for a system of particles (Raj, 2021):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0 \tag{16}$$

Where (L) is the Lagrangian of the system (a function of position and velocity), $(\dot{q}^i)$ are the generalized velocities, $(q^i)$ are the generalized coordinates (such as $(r, \phi)$). The angular momentum and energy conservation laws provide the governing principles for the particle's motion within the accretion disk and can be used to compute the detailed characteristics of the disk's structure and dynamics (Gissinger, 2024).

*Bolometric Luminosity Calculation*

In the rotating black hole system, the bolometric luminosity $(L_{\text{bol}})$ represents the total energy emitted across all electromagnetic wavelengths, providing a comprehensive measure of the system's radiative output (Bambi, 2024). This luminosity is essential for understanding the energy budget and radiative properties of the black hole and its surrounding accretion disk (Lopez, 2024). The bolometric luminosity $(L_{\text{bol}})$ is given by an integral over the spectrum of emitted radiation (Spinoglio et al., 2024). The general expression for the luminosity can be written as:

$$L_{\text{bol}} = \int_0^{\infty} \frac{dE}{dt} d\nu \tag{17}$$

where $(dE/dt)$ is the energy emitted per unit time at each frequency $(\nu)$. This is typically determined from the spectrum of the black hole, which can be modeled as a combination of various radiation processes (Synchrotron, bremsstrahlung, and thermal emission) (Hankla et al., 2022). The energy emission rate as a function of frequency $(\nu)$ depends on several factors, including the spin of the black hole, the inclination angle of the system, and the structure of the accretion disk (Pugliese & Stuchlík, 2021).

In the case of a thin accretion disk around a rotating black hole, the energy emitted across different wavelengths can be modeled using a Planck-like spectrum, with adjustments for the relativistic effects due to the black hole's spin. The differential emission rate at each frequency can be approximated as:

$$\frac{dE}{dt}(\nu) = \epsilon(\nu) \cdot \frac{r}{D^2} \cdot \left(\frac{M}{R}\right) \tag{18}$$

Where $\epsilon(\nu)$ is the spectral emissivity at frequency $(\nu)$, which depends on the temperature and density profile of the accretion disk. (r) is the radial distance from the black hole. (D) is the distance from the observer. (M) is the mass of the black hole. (R) is the Schwarzschild radius.

The temperature profile of a thin accretion disk around a rotating black hole is given by a relativistic generalization of the standard Novikov-Thorne solution for a Schwarzschild black hole (Narzilloev & Ahmedov, 2023). For a rotating black hole described by the Kerr metric, the temperature T(r) of the accretion disk at a radial coordinate (r) is (Sánchez, 2024):

$$T(r) = T_0 \left(\frac{r_{\text{ISCO}}}{r}\right)^{3/4} \tag{19}$$

Where ($T_0$) is a constant temperature normalization. ($r_{\text{ISCO}}$) is the radius of the innermost stable circular orbit (ISCO), which depends on the spin parameter (a*) of the black hole. The emissivity $\epsilon(\nu)$ is related to the temperature by the Planck function:

$$\epsilon(\nu) = \frac{16\pi h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{k_B T}} - 1} \tag{20}$$

Where (h) is Planck's constant, $(\nu)$ is the frequency of emitted radiation, (c) is the speed of light, ($k_B$) is the Boltzmann constant. After substituting the emission spectrum and the temperature profile into the expression for $(L_{\text{bol}})$, the integral can be evaluated to give a total luminosity that accounts for the energy emitted over all frequencies (Pesce et al., 2021):

$$L_{\text{bol}} = \int_0^\infty \frac{16\pi h\nu^3}{c^2} \cdot \frac{1}{e^{\frac{h\nu}{k_B T(r)}} - 1} \cdot \frac{r}{D^2} \cdot \left(\frac{M}{R}\right) d\nu \tag{21}$$

This equation encapsulates the full range of emissions from the black hole system, taking into account the relativistic modifications due to the black hole's spin, the temperature gradient in the accretion disk, and the distance between the black hole and the observer. The spin parameter (a*) plays a critical role in determining the temperature distribution and emission spectrum of the accretion disk. As the spin parameter increases, the ISCO radius decreases, leading to higher temperatures near the black hole and thus altering the spectrum of emitted radiation (Potter, 2021). This can significantly affect the luminosity, especially in the X-ray and gamma-ray bands, where relativistic effects become more pronounced. To further refine the bolometric luminosity, the frequency integral can be approximated by a combination of the disk's temperature and emissivity. This results in a more compact form of the bolometric luminosity expression.

$$L_{\text{bol}} = \frac{1}{D^2} \cdot \frac{M}{R} \cdot \int_0^\infty \frac{16\pi h\nu^3}{c^2} \cdot \frac{1}{e^{\frac{h\nu}{k_B T(r)}} - 1} d\nu \tag{22}$$

This formula provides a powerful tool to quantify the energy emitted from a rotating black hole system, taking into account relativistic corrections and the spin-dependent structure of the accretion disk. The total luminosity is a key observable, which can be directly compared with astronomical data to gain insights into the physical properties of the black hole and its surrounding environment.

*Forces on Particles in the Accretion Disk*

In the Kerr metric, the equations of motion of particles in the accretion disk are derived from the geodesic action in rotating spacetime, given by:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 \tag{23}$$

With $(\Gamma^\mu_{\alpha\beta})$ is the Christoffel coefficient which depends on the rotation parameter (a) and mass (M) of the black hole. The total energy (E) and azimuthal angular momentum ($L_z$) for a particle moving in a stable circular orbit in the Kerr metric are given by:

$$E = \frac{1 - \frac{2M}{r} + a\Omega}{\sqrt{1 - \frac{3M}{r} + 2a\Omega}} \tag{24}$$

$$L_z = \frac{(r^2 - 2Mr + a^2)\Omega - a}{\sqrt{1 - \frac{3M}{r} + 2a\Omega}} \tag{25}$$

with $(\Omega)$ as the particle orbital frequency:

$$\Omega = \frac{M^{1/2}}{r^{3/2} + aM^{1/2}} \tag{26}$$

The total force acting on particles in the accretion disk comes from a combination of relativistic gravity, centrifugal force, and frame-dragging effects due to the black hole's rotation:

$$F_r = -\frac{dV_{\text{eff}}}{dr} \tag{27}$$

with effective potential:

$$V_{\text{eff}}(r,a,L_z) = \frac{1}{2}\left(1 - \frac{2M}{r} + \frac{aL_z}{r^3}\right) \tag{28}$$

If this force is expressed in logarithmic form to describe the non-linear effects in the near-disk region:

$$F_r = -\frac{d}{dr}\left[\ln\left(\frac{1 - \frac{2M}{r} + \frac{aL_z}{r^3}}{1 - \frac{3M}{r} + 2a\Omega}\right)\right] \tag{29}$$

The spin of the black hole (a* = a/M) affects the orbital stability and energy distribution in the accretion disk. The value of (x) as a parameter relative to ISCO (inner stable circular orbit) is given by:

$$x_0 = \sqrt{\frac{r_{\text{ISCO}}}{M}} \tag{30}$$

With the explicit solution of ISCO as a function of spin:

$$r_{\text{ISCO}} = M\left[3 + Z_2\ \text{m}\sqrt{(3 - Z_1)(3 + Z_1 + 2Z_2)}\right] \tag{31}$$

$$Z_1 = 1 + \left(1 - a_*^2\right)^{1/3}\left[\left(1 + a_*\right)^{1/3} + \left(1 - a_*\right)^{1/3}\right] \tag{32}$$

$$Z_2 = \sqrt{3a_*^2 + Z_1^2} \tag{33}$$

To understand the force variations in the accretion disk, we use a logarithmic scale to see the significant differences near ISCO.:

$$\log F(x) = A\log(x - x_0) + B\sin(Cx) \tag{34}$$

with (A, B, C) are constants that depend on the system parameters (mass, angular momentum, and spin). The radial force in the accretion disk shows a non-linear dependence on the spin of the black hole (a*). For high spin $(a_* \sim 1)$, the maximum force occurs at a smaller radius, causing the disk to be more compact and increasing the relativistic effect. For low spin $(a_* \sim 0)$, forces are more evenly distributed, causing the disk to be wider and similar to the Schwarzschild structure.

*Spin Parameter (a) and Its Effect on Accretion Disk Structure*

The innermost stable circular orbit (ISCO) in the accretion disk depends on the $(a^*)$. The minus sign applies to prograde orbits $(a^* > 0)$, and the plus sign to retrograde orbits $(a^* < 0)$. When $(a^* \to 1)$, $(r_{\text{ISCO}})$ approaches (M), which means the material can be very close to the event horizon without losing its orbital stability. In Boyer-Lindquist coordinates, the effective potential for particles orbiting a Kerr black hole is:

$$V_{\text{eff}}(r) = -\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GMLa^*}{c^2r^3} \tag{35}$$

The relativistic orbital velocity in the accretion disk also depends on $(a^*)$, with the expression:

$$v_\phi = \frac{c}{\sqrt{1 - \frac{3M}{r} + 2a^*\left(\frac{M}{r}\right)^{3/2}}} \tag{36}$$

The larger $(a^*)$, the faster the rotation of the material in the accretion disk. This also causes a significant temperature difference between the inside and outside of the disk, affecting the spectrum of the emitted radiation. The efficiency of mass-to-radiation conversion in an accretion disk is determined by:

$$\eta = 1 - E_{\text{ISCO}} \tag{37}$$

with $(E_{\text{ISCO}})$ as the specific energy of the ISCO orbit, which in the case of Kerr black holes is given by:

$$E_{\text{ISCO}} = \frac{1 - \frac{2}{3r_{\text{ISCO}}} + a^*\left(\frac{2}{3r_{\text{ISCO}}}\right)^{3/2}}{\sqrt{1 - \frac{3}{r_{\text{ISCO}}} + 2a^*\left(\frac{1}{r_{\text{ISCO}}}\right)^{3/2}}} \tag{38}$$

This efficiency increases drastically for black holes with $(a^* \approx 1)$, where $(\eta)$ can reach about 42%, compared to only 6% for Schwarzschild black holes $(a^* = 0)$. This suggests that black holes with high rotation can be more efficient at converting gravitational energy into electromagnetic radiation. The maximum temperature in the accretion disk can be estimated by:

$$T_{\max} \approx \left( \frac{3GM\dot{M}}{8\pi\sigma r_{\mathrm{ISCO}}^{3}} \right)^{1/4} \tag{39}$$

where $(\dot{M})$ is the mass accretion rate and $(\sigma)$ is the Stefan-Boltzmann constant. As $(r_{\mathrm{ISCO}})$ decreases at $(a^{*} \to 1)$, the temperature of the accretion disk tends to be higher, causing more emission in the X-ray spectrum. The angular momentum distribution in the relativistic accretion disk around a Kerr black hole can be expressed as:

$$L_z = \frac{GMr^{1/2}}{c} \frac{1-2a^{*}(r/M)^{-3/2} + a^{*2}(r/M)^{-2}}{\sqrt{1-3(r/M)^{-1} + 2a^{*}(r/M)^{-3/2}}} \tag{40}$$

Changes in $(a^{*})$ cause variations in the angular momentum distribution, which affects the thickness of the disk and the stability of the matter flow. The rotation of the black hole affects the structure of the accretion disk, with ISCO smaller for $(a^{*} > 0)$, allowing matter to approach the event horizon before stability is lost. The orbital velocity and disk temperature increase with $(a^{*})$, causing changes to the emission spectrum and energy distribution in the accretion system. The radiation efficiency increases with $(a^{*})$, with the maximum efficiency reaching 42% for $(a^{*} = 1)$, which is significantly higher than that of a Schwarzschild black hole. The angular momentum distribution in the accretion disk strongly depends on $(a^{*})$, which affects the orbital stability and launch mechanism of the relativistic jet. The influence of $(a^{*})$ on the disc structure can be observed through thermal effects and material dynamics, which has implications in X-ray modeling of quasars and active black holes.

## Method

*Algorithm for Calculating the Force F(x) on Particles in an Accretion Disk Around a Rotating Black Hole Using Spin Parameters and Relative Position*

The algorithm is used to calculate the forces acting on particles in the accretion disk around a rotating black hole, taking into account several physical parameters such as the spin parameter $(a*)$ as well as other parameters such as $(x_0)$, $(x_1)$, $(x_2)$, and $(x_3)$. The main function used is F(x), which calculates the force based on the relative position of the object in the gravitational field of the black hole. The function F(x) accepts five inputs: (x), $(a*)$, $(x_0)$, $(x_1)$, $(x_2)$, and $(x_3)$. Each parameter describes a variable that relates to the object's relative position in the black hole's gravitational field as well as other parameters that affect the forces on the object.

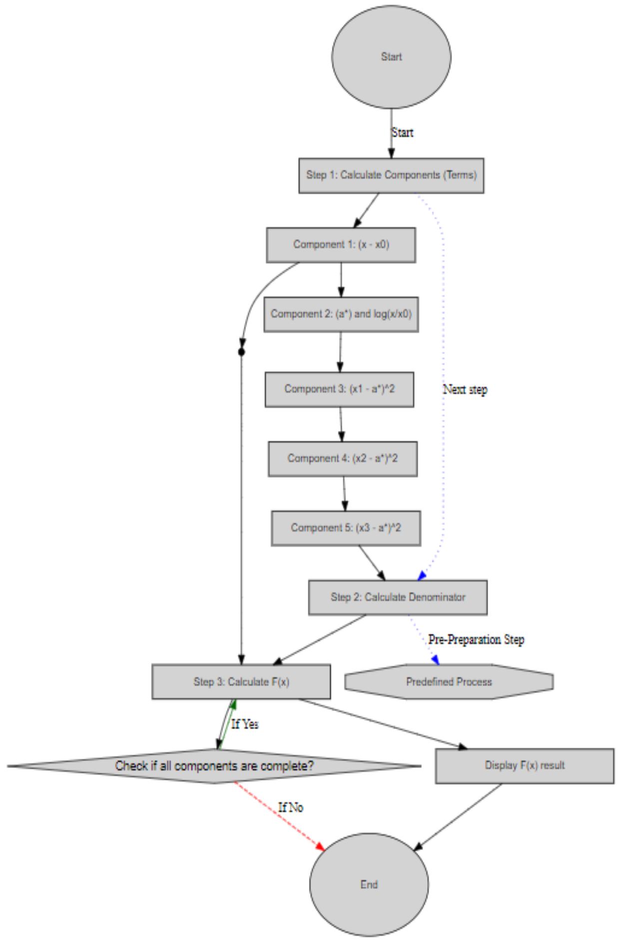


**Figure 1.** A structured calculation process involving three main steps

Figure 1 is a flowchart outlining the structured calculation process that involves three main steps: 1) calculating the components (terms), 2) calculating the denominator, and 3) calculating the final result, F(x). The process includes iterative steps with decision points to check if all components are complete before proceeding. If they are complete, the result will be displayed; otherwise, the process will end. There are predefined preparatory steps and clear transitions between the stages, highlighting an orderly flow from start to finish.

Figure 2 is an algorithm to calculate the roots of a cubic equation based on the parameter $(a*)$. It starts with an input value of $(a*)$ in the range [-1, 1], then checks if the value is within the valid range. If yes, the algorithm continues the calculation to get the three roots of $(x_1)$, $(x_2)$, and $(x_3)$ using the trigonometric formula. After that, the results are saved and the process is terminated. If the value of $(a*)$ is not within the valid range, the process terminates immediately.

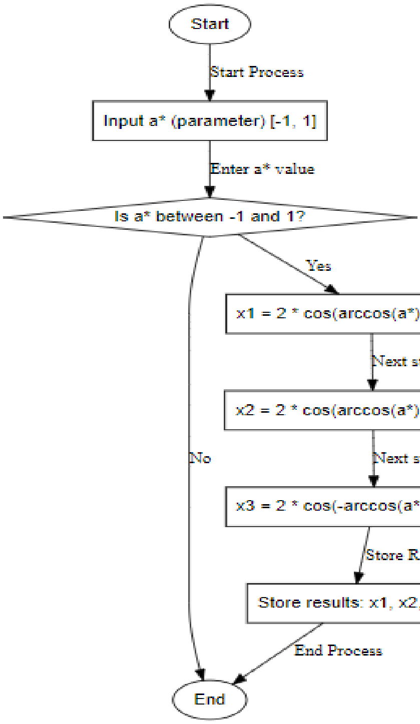


**Figure 2.** Algorithm for calculating the roots of a cubic equation based on the parameter (a*)

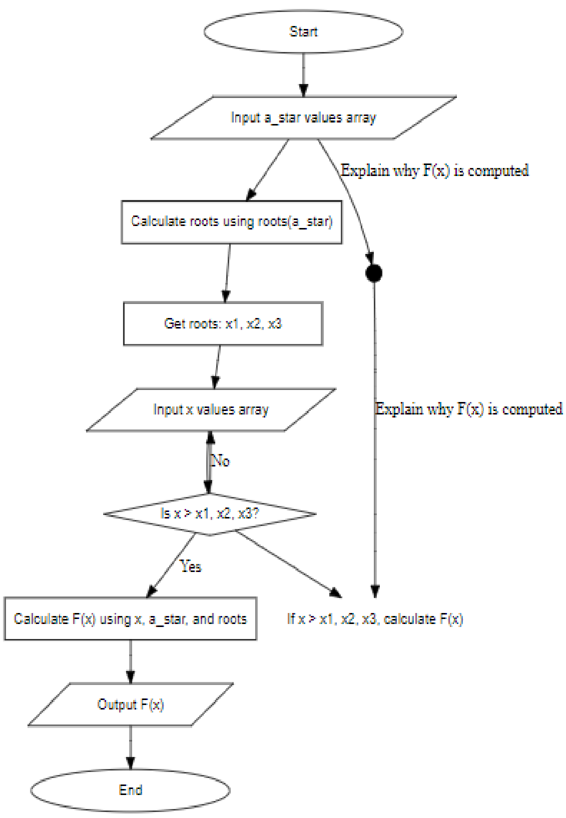


**Figure 3.** Algorithm to calculate the value of F(x) based on the input array `a_star` and the value `x`

The value of the rotation parameter $(a*)$ is defined in the range from 0 to 0.99 with an interval of 0.2 to describe the variation of the black hole rotation speed in the analysis. The radius value (x) is defined in the range from 1.5 to 10 with an interval of 0.1 to describe the relative position of objects around the black hole.

Figure 3 is the algorithm to calculate the value of F(x) based on the input array `a_star` and the value `x`. The process starts with inputting the value of `a_star`, then calculates the roots of the cubic equation for `a_star` and gets the root value of `x` ($x_1$, $x_2$, $x_3$). After that, the value of `x` is input and compared with those roots to determine if it is greater than one of the roots. If it is, F(x) is calculated using the `x`, `a_star`, and the roots obtained. The result of the F(x) calculation is then output and the process ends.

*Calculation of Forces on Particles in an Accretion Disk Around a Rotating Black Hole Based on Energy, Angular Momentum, and Spin Parameters, with Logarithmic and Trigonometric Function Approaches*

In the Kerr metric, which describes the spacetime around a rotating black hole, the time-averaged radial structure of the accretion disk can be expressed through an equation involving several important parameters such as total energy (E), angular momentum ($L_z$), and rotation frequency $(\Omega)$ (Röder et al., 2023; Sánchez, 2024). The bolometric luminosity, which is the total electromagnetic luminosity of an object, is calculated by integrating the luminosity across wavelengths (Hogg, 2022; Saccheo et al., 2023). In a thin accretion disk, the relationship between these parameters can be described by the following equation (Abramowicz & Fragile, 2013; Stashko et al., 2021):

$$F(x) = -\frac{\partial_r \Omega}{(E-\Omega L_z)^2} \frac{M^2}{\sqrt{-G}} \int_{r_{in}}^{r} (E-\Omega L_z)(\partial_x L_z)\, dx \quad (41)$$

This equation represents the relationship between the force F(x) acting on a particle in an accretion disk and other parameters such as (E), $(\Omega)$, and ($L_z$). To accommodate more complex models, this equation can be further developed into a form involving logarithmic and trigonometric functions. Thus, the more complicated form of F(x) is as follow:

$$F(x)=\frac{3}{2}\frac{1}{x^4(x^3-3x+2a_*)}\left\{x-x_0-\frac{3}{2}a_*\ln\left(\frac{x}{x_0}\right)-\frac{3(x_1-a_*)^2}{x_1(x_1-x_2)(x_1-x_3)}\ln\left(\frac{x-x_1}{x_0-x_1}\right)-\frac{3(x_2-a_*)^2}{x_2(x_2-x_1)(x_2-x_3)}\ln\left(\frac{x-x_2}{x_0-x_2}\right)\right.$$
$$\left.-\frac{3(x_3-a_*)^2}{x_3(x_3-x_1)(x_3-x_2)}\ln\left(\frac{x-x_3}{x_0-x_3}\right)\right\} \quad (42)$$

In this equation, ($x_0$) is defined as the square root of the ratio between ($r_{\text{isco}}$) (the radius of the nearest stable orbit or ISCO) and the mass (M) of the black hole. This ($x_0$) parameter describes the characteristic length associated with the orbits of objects close to the black hole. It also indicates the critical distance around the black hole where particles begin to lose their orbital stability (Pretorius & Khurana, 2007). It is a very important parameter in determining the region of orbital stability in the extreme gravitational field around a black hole:

$$x_0=\sqrt{\frac{r_{\text{isco}}}{M}} \quad (43)$$

$$x=\sqrt{\frac{r}{M}} \quad (44)$$

The parameter (x) itself is the square root of the ratio between the general radius (r) and the mass (M), which indicates the size relative to the mass of the black hole. This describes the relative position of the object in the black hole's gravitational field.

The values ($x_1$), ($x_2$), and ($x_3$) are solutions of a cubic equation involving the spin parameter $(a*)$ of the black hole, calculated using the trigonometric cosine function. These values are obtained by applying the inverse cosine to the spin parameter $(a*)$, which is a decimal number value between -1 and 1, describing the rotational speed of the black hole. By considering the solutions of these cubic equations, the values of ($x_1$), ($x_2$), and ($x_3$) can be calculated and describe the gravitational properties associated with the rotating black hole, as well as its effect on the structure of the accretion disk:

$$x_1=2\cos\left\{\frac{1}{3}\arccos(a_*)-\frac{\pi}{3}\right\} \quad (45.a)$$

$$x_2=2\cos\left\{\frac{1}{3}\arccos(a_*)+\frac{\pi}{3}\right\} \quad (45.b)$$

$$x_3=-2\cos\left\{\frac{1}{3}\arccos(a_*)\right\} \quad (45.c)$$

All of these equations representing the forces acting on particles in the accretion disk around the spinning black hole can be calculated by considering various parameters, including the energy, angular momentum, and spin of the black hole.

## Result and Discussion

*Effect of Spin Parameters on Singularity Phenomenon in Black Holes and the Limit of No Singularity at a Certain Radius*

In this study, the phenomenon of significant changes in black holes that depend on the value of the spin parameter $a^*$ is observed. This significant change does not occur at a certain value of radius, referred to as the limit without significant change, which can be seen at various values of $a^*$. As seen at values of $a^*=0.6$, there is no significant change up to a radius of (x = 1.7), where the value of F(x) is recorded to be about 0.0086 µ. At ($a^*=0.4$), the boundary without significant change occurs up to (x=2), with a F(x) value of 610.5 µ. Similarly, for ($a^*=0.2$), there is no significant change until the point (x = 2.2), with the value of F(x) reaching 95.02 µ. As for ($a^*=0$), significant changes only appear at radius (x = 2.2), with a F(x) value of about 321.54 µ.

**Table 1.** Relationship of Spin Parameter with Point of No Singularity and Point of Return Singularity on Black Hole Accretion Disk

| Spin Parameters | Point Without Significant Change | Significant Change Point Back |
|---|---|---|
| 0.6 | (1.7, 0.0086 µ) | (2.3, 208.06 µ) |
| 0.4 | (2.0, 610.5 µ) | (2.3, 104.07 µ) |
| 0.2 | (2.2, 95.02 µ) | (2.4, 10.91 µ) |
| 0 | (2.2, 321.54 µ) | (2.8, 51.73 µ) |
| 0.8 | - | (2.3, 278.84 µ) |

However, at higher values of rotation, significant changes reappear at larger radii. At ($a^*=0.8$), significant changes reappear at (x = 2.3), with a F(x) value of 278.84 µ. Similarly, for ($a^*=0.6$), significant changes appear at the point (x = 2.3) with a value F(x) of about 208.06 µ, and for ($a^*=0.4$), significant changes appear at (x = 2.3) with a value $F(x)$ = 104.07 $\mu$. For ($a^*=0.2$), a significant change occurs at (x = 2.4), with a value of F(x) = 10.91 µ, and at ($a^*=0$), a significant change appears at the point (x = 2.8) with a value of F(x) = 51.73 µ.

The study expands our understanding of the dynamics of black hole accretion disks by showing points of no significant change and points of significant change at various values of the spin parameter (a*). This confirms how structural changes in the accretion disk occur, which is closely related to existing accretion disk theories. In theory, physical parameters such as the spin of the black hole affect the structure and dynamics of the matter flow, including the velocity and temperature distributions in the disk (Reynolds, 2021).

The value of the spin parameter ($a^*$) affect the temperature and velocity distribution, and determine the location of significant change points on the accretion disk (Bisnovatyi-Kogan & Lovelace, 2001). This result is consistent with the predictions of accretion disk theory, which suggests that materials in the disk can undergo phase changes or transitions influenced by the spin of the black hole, causing significant changes in the physical properties of the disk (Abramowicz & Fragile, 2013).

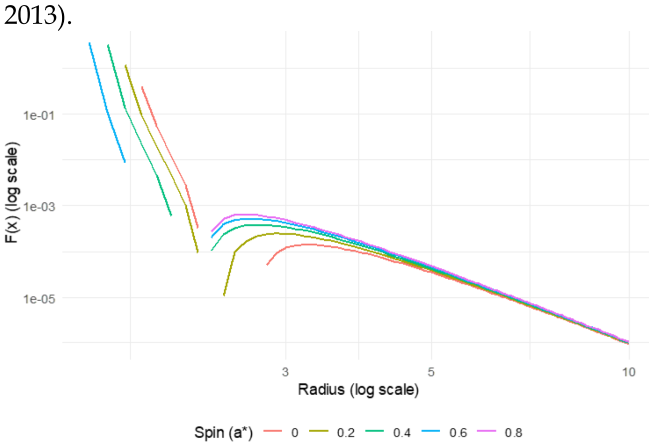


**Figure 5.** Relationship between radius and F(x) values for various values of spin parameter

Figure 5 shows the relationship between the radius (x) and the value of F(x) for various values of the spin parameter ($a^*$), which affects the accretion disk structure of the black hole. In the graph, it can be seen that F(x) depends on the value of the spin parameter ($a^*$), where the value of F(x) for each value of ($a^*$) records a significant change as the radius (x) changes. As the value of the spin parameter increases, the F(x) function shows sharper variations around a certain point. This indicates that the accretion disk structure of the black hole is strongly influenced by the spin value of the black hole (Abramowicz & Fragile, 2013). Based on these results, it has been observed that for some specific spin values, there is a limit point where there is no significant change until the radius (x) reaches a certain point. At lower spin values (such as $a^* = 0.2$), significant changes do not appear until the radius (x = 2.2). Whereas at higher values of rotation (such as $a^* = 0.8$), significant changes begin to appear at smaller radii (such as x = 2.3).

The use of a logarithmic scale on the F(x) graph aims to clarify the fluctuations in the value of F(x) over the entire range of the accretion disk radius. In many cases, small changes in the function F(x) at distances very close to the center of the black hole can be very significant and can only be clearly seen when using a logarithmic scale. This makes it possible to see large and small changes in the function that involve extreme variations, which are not obvious when using a regular linear scale. Logarithmic scales are used to describe distributions that involve very different values or have large differences between data points, such as in the distribution of matter in an accretion disk that shows sharp changes in energy or temperature at a certain distance from the center of the black hole. The use of logarithms facilitates the visualization and interpretation of data that has a large range, so that the relationship between data points further and closer to the black hole can be effectively compared.

## Conclusion

The force F(x) is strongly influenced by the combination of total energy (E), angular momentum, and spin parameter, which shows the complex interaction between the accretion disk and the gravitational field of the black hole. The spin parameter determines the location of the stable radius (ISCO) and affects the radial structure of the accretion disk. The larger the spin value, the more significant the change in the force F(x) at a given radius. The cubic root solution describes the limit of no singularity and the return of the singularity at a certain radius. The use of logarithmic and trigonometric functions extends the force model, interpreting the complicated space-time dynamics around a spinning black hole. Theoretically, the algorithm provides a quantitative approach to understanding the distribution of forces on particles in accretion disks, which is relevant for the study of extreme gravity and black hole dynamics. The algorithm can be used to predict the force distribution in the accretion disc, which is useful in analyzing the electromagnetic radiation patterns of active black holes. The model helps determine the radius without singularities for a given spin value, which is important for understanding the stability of particle orbits. Understanding the relationship between spin, force, and disk structure can improve the interpretation of observational data, such as emission spectra from black holes. The algorithm can be implemented in computational simulations to model accretion disk systems more accurately, which is relevant for the study of black hole evolution and matter accretion.

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**Author Contributions**

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**Conflicts of Interest**

The authors declare no conflict of interest.

## References

Abramowicz, M. A., & Fragile, P. C. (2013). Foundations of black hole accretion disk theory. *Living Reviews in Relativity*, *16*, 1–88. https://doi.org/10.12942/lrr-2013-1

Anitra, A. (2024). *Accretion disc tomography in Low-Mass X-Ray Binaries: A multi-wavelength approach*. Retrieved from https://hdl.handle.net/10447/624153

Ashoorioon, A., Poshteh, M. B. J., & Luongo, O. (2024). Particle motion and accretion disk around rotating accelerating black holes. *arXiv Preprint arXiv:2408.07762*. https://doi.org/10.48550/arXiv.2408.07762

Baehr, H., & Zhu, Z. (2021). Particle Dynamics in 3D Self-gravitating Disks. II. Strong Gas Accretion and Thin Dust Disks. *The Astrophysical Journal*, *909*(2), 136. https://doi.org/10.3847/1538-4357/abddb4

Bambi, C. (2024). Black hole X-ray spectra: Notes on the relativistic calculations. *arXiv Preprint arXiv:2408.12262*. https://doi.org/10.48550/arXiv.2408.12262

Banerjee, P., Herrera-Aguilar, A., Momennia, M., & Nucamendi, U. (2022). Mass and spin of Kerr black holes in terms of observational quantities: The dragging effect on the redshift. *Physical Review D*, *105*(12), 124037. https://doi.org/10.48550/arXiv.2203.09092

Béthune, W., Latter, H., & Kley, W. (2021). Spiral structures in gravito-turbulent gaseous disks. *Astronomy & Astrophysics*, *650*, A49. https://doi.org/10.1051/0004-6361/202040094

Bisnovatyi-Kogan, G., & Lovelace, R. (2001). Advective accretion disks and related problems including magnetic fields. *New Astronomy Reviews*, *45*(11–12), 663–742. https://doi.org/10.1016/S1387-6473(01)00146-4

Boruah, A. J., Devi, L., & Sarkar, B. (2024). Relativistic Accretors and High Energy X-Ray view. *arXiv Preprint arXiv:2410.12230*. https://doi.org/10.48550/arXiv.2410.12230

Bourne, M. A., Fiacconi, D., Sijacki, D., Piotrowska, J. M., & Koudmani, S. (2024). Dynamics and spin alignment in massive, gravito-turbulent circumbinary discs around supermassive black hole binaries. *Monthly Notices of the Royal Astronomical Society*, *534*(4), 3448–3477. https://doi.org/10.1093/mnras/stae2143

Cattorini, F., & Giacomazzo, B. (2024). GRMHD study of accreting massive black hole binaries in astrophysical environment: A review. *Astroparticle Physics*, *154*, 102892. https://doi.org/10.1016/j.astropartphys.2023.102892

Crinquand, B. (2021). *Particle acceleration in Kerr black hole magnetospheres*. Retrieved from https://inspirehep.net/literature/2030518.

Fung, P. C., & Wong, K. (2015). On the Origin of Mass and Angular Momentum of Stellar Objects. *Journal of Modern Physics*. http://doi.org/10.4236/jmp.2015.615235

Gimeno-Soler, S., Font, J. A., Herdeiro, C., & Radu, E. (2021). Magnetized accretion disks around Kerr black holes with scalar hair: Nonconstant angular momentum disks. *Physical Review D*, *104*(10), 103008., https://doi.org/10.1103/PhysRevD.104.103008

Gissinger, C. (2024). Laboratory modeling of MHD accretion disks. *Comptes Rendus. Physique*, *25*(S3), 1–28. https://doi.org/10.5802/crphys.204

Hankla, A. M., Scepi, N., & Dexter, J. (2022). Non-thermal emission from the plunging region: A model for the high-energy tail of black hole X-ray binary soft states. *Monthly Notices of the Royal Astronomical Society*, *515*(1), 775–784. https://doi.org/10.1093/mnras/stac1785

Hogg, D. W. (2022). Magnitudes, distance moduli, bolometric corrections, and so much more. *arXiv Preprint arXiv:2206.00989*. https://doi.org/10.48550/arXiv.2206.00989

Hou, Y., Zhang, Z., Yan, H., Guo, M., & Chen, B. (2022). Image of a Kerr-Melvin black hole with a thin accretion disk. *Physical Review D*, *106*(6), 064058. https://doi.org/10.1103/PhysRevD.106.064058

Kagohashi, E., Suzuki, R., & Tomizawa, S. (2024). Innermost stable circular orbits around a spinning black hole binary. *Physical Review D*, *110*(2), 024013. https://doi.org/10.1103/PhysRevD.110.024013

Kostić, U., Čadež, A., Calvani, M., & Gomboc, A. (2009). Tidal effects on small bodies by massive black holes. *Astronomy & Astrophysics*, *496*(2), 307–315., https://doi.org/10.1051/0004-6361/200811059

Kraniotis, G. (2021). Gravitational redshift/blueshift of light emitted by geodesic test particles, frame-dragging and pericentre-shift effects, in the Kerr–Newman–de Sitter and Kerr–Newman black hole geometries. *The European Physical Journal C*, *81*(2), 1–37. https://doi.org/10.1140/epjc/s10052-021-08911-5

Li, Z., & Bambi, C. (2014). Measuring the Kerr spin parameter of regular black holes from their shadow. *Journal of Cosmology and Astroparticle Physics*, *2014*(01), 041. https://doi.org/10.1088/1475-7516/2014/01/041

Lopez, I. E. (2024). *Accretion properties of supermassive black holes across cosmic time and luminosities*. Retrieved from http://amsdottorato.unibo.it/id/eprint/11583

Maes, S. H. (2023). *Dynamic sources, Dynamic Multi-folds, and General Relativity Lense-Thirring and Frame Dragging Effects*., Retrieved from https://shmaesphysics.wordpress.com/2023/03/12/dynamic-sources-dynamic-multi-folds-and-general-relativity-lens-thirring-and-frame-dragging-effects/

Malki, Z. (2024). *Theoretical Study Of Physical Phenomena Around A ProtoPlanetary Disk*. Retrieved from https://dspace.univ-ouargla.dz/jspui/handle/123456789/35522

Middleton, M. (2016). Black hole spin: Theory and observation. *Astrophysics of Black Holes: From Fundamental Aspects to Latest Developments*, 99–151. https://doi.org/10.48550/arXiv.1507.06153

Morales-Herrera, G., Ortega-Ruiz, P., Momennia, M., & Herrera-Aguilar, A. (2024). Mass, charge, and distance to Reissner–Nordström black hole in terms of directly measurable quantities. *The European Physical Journal C*, *84*(5), 525. https://doi.org/10.1140/epjc/s10052-024-12880-w

Narzilloev, B., & Ahmedov, B. (2023). Thermal radiation of thin accretion disk around Taub-NUT black hole. *International Journal of Modern Physics D*, *32*(10), 2350064. https://doi.org/10.1142/S0218271823500645

O'Neill, B. (2014). *The geometry of Kerr black holes*. Courier Corporation. Retrieved from https://www.scirp.org/reference/referencespapers?referenceid=745371

Pesce, D. W., Palumbo, D. C., Narayan, R., Blackburn, L., Doeleman, S. S., Johnson, M. D., Ma, C.-P., Nagar, N. M., Natarajan, P., & Ricarte, A. (2021). Toward determining the number of observable supermassive black hole shadows. *The Astrophysical Journal*, *923*(2), 260. https://doi.org/10.48550/arXiv.2108.05228

Potter, W. J. (2021). A full relativistic thin disc–the physics of the plunging region and the value of the stress at the ISCO. *Monthly Notices of the Royal Astronomical Society*, *503*(4), 5025–5045. https://doi.org/10.1093/mnras/stab636

Pugliese, D., & Stuchlík, Z. (2021). General relativistic rotational energy extraction from black holes-accretion disk systems. *Classical and Quantum Gravity*, *38*(14), 145014. https://doi.org/10.1088/1361-6382/abff97

Punsly, B. (2008). *Black hole gravitohydromagnetics* (Vol. 355). Springer Science & Business Media. https://doi.org/10.1007/978-3-540-76957-6

Raj, A. (2021). *Dynamic accretion discs around supermassive black holes*. Retrieved from https://figshare.le.ac.uk/articles/thesis/Dynamic_accretion_discs_around_supermassive_black_holes/14518818?file=27823179

Reynolds, C. S. (2021). Observational constraints on black hole spin. *Annual Review of Astronomy and Astrophysics*, *59*(1), 117–154. https://doi.org/10.48550/arXiv.2011.08948

Röder, J., Cruz-Osorio, A., Fromm, C. M., Mizuno, Y., Younsi, Z., & Rezzolla, L. (2023). Probing the spacetime and accretion model for the Galactic Center: Comparison of Kerr and dilaton black hole shadows. *Astronomy & Astrophysics*, *671*, A143. https://doi.org/10.48550/arXiv.2301.09549

Rossi, E. M., Stone, N. C., Law-Smith, J. A., MacLeod, M., Lodato, G., Dai, J. L., & Mandel, I. (2021). The Process of Stellar Tidal Disruption by Supermassive Black Holes: The First Pericenter Passage. *Space Science Reviews*, *217*, 1–48. https://doi.org/10.48550/arXiv.2005.12528

Saccheo, I., Bongiorno, A., Piconcelli, E., Testa, V., Bischetti, M., Bisogni, S., Bruni, G., Cresci, G., Feruglio, C., & Fiore, F. (2023). The WISSH quasars project-XI. The mean spectral energy distribution and bolometric corrections of the most luminous quasars. *Astronomy & Astrophysics*, *671*, A34. https://doi.org/10.1051/0004-6361/202244296

Sánchez, L. A. (2024). Quantum gravity modifications to the accretion onto a Kerr black hole. *The European Physical Journal C*, *84*(6), 635. Retrieved from https://link.springer.com/article/10.1140/epjc/s10052-024-12974-5

Schinnerer, E., & Leroy, A. K. (2024). Molecular Gas and the Star-Formation Process on Cloud Scales in Nearby Galaxies. *Annual Review of Astronomy and Astrophysics*, *62*. Retrieved from https://arxiv.org/abs/2403.19843

Sheoran, P., Herrera-Aguilar, A., & Nucamendi, U. (2018). Mass and spin of a Kerr black hole in modified gravity and a test of the Kerr black hole hypothesis. *Physical Review D*, *97*(12), 124049. https://doi.org/10.1103/PhysRevD.97.124049

Siagian, R. C., Alfaris, L., Muhammad, A. C., Nyuswantoro, U. I., & Rancak, G. T. (2023). The Orbital Properties of Black Holes: Exploring the Relationship between Orbital Velocity and Distance. *Journal of Physics and Its Applications*, *5*(2), 35–42. https://doi.org/10.14710/jpa.v5i2.17860.

Spinoglio, L., Fernández-Ontiveros, J. A., & Malkan, M. A. (2024). The Spectral Energy Distributions and Bolometric Luminosities of Local AGN: Study of the Complete 12 μm AGN Sample. *The Astrophysical Journal*, *964*(2), 117. Retrieved from https://iopscience.iop.org/article/10.3847/1538-4357/ad23e4.

Stashko, O., Zhdanov, V., & Alexandrov, A. (2021). Thin accretion discs around spherically symmetric configurations with nonlinear scalar fields. *Physical Review D*, *104*(10), 104055. https://doi.org/10.1103/PhysRevD.104.104055

Teukolsky, S. A. (2015). The kerr metric. *Classical and Quantum Gravity*, *32*(12), 124006., Teukolsky, S. A. (2015). The kerr metric. Classical and Quantum Gravity, 32(12), 124006. https://doi.org/10.1088/0264-9381/32/12/124006

Turimov, B., Davlataliev, A., Usmanov, Y., Karshiboev, S., & Tadjimuratov, P. (2024). Investigating new aspects of Bocharova–Bronnikov–Melnikov–Bekenstein spacetime: Ionized thin accretion disk model. *The European Physical Journal C*, *84*(10), 1–10. https://doi.org/10.1140/epjc/s10052-024-13426-w

Volonteri, M., Habouzit, M., & Colpi, M. (2021). The origins of massive black holes. *Nature Reviews Physics*, *3*(11), 732–743. https://doi.org/10.48550/arXiv.2110.10175

Wessel, E., Paschalidis, V., Tsokaros, A., Ruiz, M., & Shapiro, S. L. (2021). Gravitational waves from disks around spinning black holes: Simulations in full general relativity. *Physical Review D*, *103*(4), 043013. https://doi.org/10.1103/PhysRevD.103.043013