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Cofactor Expansion Approaches for Generalized Determinants of Non-Square Matrices

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Abstract: Determinants are conventionally defined only for square matrices, which leaves the theoretical discussion of rectangular matrices relatively undeveloped. This study aims to extend determinant theory by introducing a generalized form of the cofactor expansion applicable to non-square matrices. The method employed in this research is a theoretical–analytical approach that begins with identifying fundamental determinant axioms and proceeds with the construction of a recursive cofactor expansion for $m \times n$ matrices with $m \leq n$. Worked examples are used to demonstrate the linearity, antisymmetry under row interchange, and reduction consistency of the proposed formulation when applied to square matrices. The findings indicate that the generalized expansion preserves essential algebraic properties and shows compatibility with established rectangular determinant definitions, including the Radić determinant. Overall, this study provides a coherent theoretical foundation for the generalized determinant of rectangular matrices and contributes to broader applications of determinant-based analysis in linear algebra.

Keywords: Cofactor expansion; Generalized determinant; Linear algebra; Non-square matrix; Radic determinant

Introduction

Mathematics is the language of all sciences. It offers a structured approach to describing and interpreting phenomena of all types. These include the natural and social sciences as well as computing phenomena (Stewart et al., 2016; Kolman et al., 2008). Linear algebra is certainly among the most influential and useful disciplines of mathematics because it assists the theory and practice of multidimensional system modeling in physics, economics, engineering, and data science (Anton et al., 2014; Strang et al., 2019). Matrices are fundamental in the representation and manipulation of linear transformations. Determinants, on the other hand, serve as scalars that summarize the contours of a matrix concerning invertibility, eigenvalues, and orientation transformations (Leon et al., 2010; Poole et al., 2015). Within the mathematics of linear equation systems, geometry (area and volume) and Jacobian evaluation, the computation of determinants is a critical part (Kharie et al., 2021; Makarewicz et al., 2016).

The determinant is classically defined for square matrices only- matrices that have the same number of rows and columns. There are a number of classical techniques for determinant calculation, such as the Sarrus rule, the row-reduction method, and Laplace's cofactor expansion (Hill et al., 2008; DeFranza et al., 2009; Leon et al., 2010). When it comes to a non-square matrix, however, the standard definitions become mathematically invalid as the diagonals needed to evaluate the determinant are absent (Mursaid et al., 2018). This issue is not new, and many mathematicians have tried to extend the definitions of the determinant to include matrices that are not square (Cullis at al., 1913; Radić et al., 1966; Joshi et al., 1980).

In his 1913 work Matrices and Determinoids, Cullis was the first to generalize a determinant to rectangular matrices by proposing the idea of defining a determinant as an algebraic combination of equal order minors.

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(Radić et al, 1966) built upon this idea and formalized what we now refer to as the Radić determinant by representing a rectangular determinant as a sum of the determinants of its square submatrices. This idea became the foundation for further research and reinterpretation (Joshi et al., 1980; Stanimirović et al., 1997; Arunkumar et al., 2011; Abdollahi et al., 2015; Yurchenko et al., 2023). The computational methods proposed by Radić, as well as those that followed him, lack practicality, especially for large matrices, due to the algebraic complications that arise (Amiri et al., 2010; Fitri et al., 2018; Putra et al., 2020).

Apart from computational issues, some previous work has not adequately addressed whether all basic properties of square determinants, such as linearity, antisymmetry under row exchange, and multiplicative consistency, also apply to non-square matrices (Nur et al., 2014; Makarewicz et al., 2014; Amiri et al., 2011). These issues indicate that, although there are theoretical foundations involving non-square determinants, the algebraic extension and rational coherence of the properties of determinants are not fully developed. Also, most of the existing literature does not address how the cofactor expansion method, fundamental to the theory of determinants, can be generalized to work with rectangular matrices. This poses an important theoretical gap in the complete theory of determinants and their use in solving linear systems and in the higher order analysis.

The current research aims to fill this gap by introducing a generalized cofactor expansion for defining determinants over non-square matrices. The approach generalizes the concept of the cofactor expansion for square matrices to rectangular matrices. The process aims to maintain determinant features, such as linearity, anti-symmetry and scalar multiplication, while the research is different from the Radić approach, which requires numerous summations of the sub-matrix determinants. This researched approach is more structured and involves a recursive expansion process that simplifies computational processes but retains theoretical soundness. Hence, the purpose of this research lies in the following: (1) the formulation of a generalized cofactor expansion applicable to non-square matrices, (2) the demonstration that essential determinant properties remain valid under this extension, and (3) the establishment of a more straightforward computational framework compared to existing techniques.

Recent studies, such as (Zhang et al., 2024), have extended determinant concepts to tensor and higher-dimensional matrix structures. However, these works primarily focus on computational models rather than the algebraic consistency of determinant properties, leaving theoretical generalization gaps that this study aims to address.

Through the introduction of this generalized framework, the present study advances the unification of determinant concepts applicable to both square and rectangular matrices. This development reinforces the algebraic underpinnings of linear algebra and broadens its applicability across various domains, including numerical analysis, data transformation, and theoretical modeling (Abdollahi et al., 2015; Yurchenko et al., 2023; Radić et al., 2008).

Therefore, the purpose of this research is to formulate a generalized cofactor expansion applicable to non-square matrices, demonstrate that essential determinant properties remain valid under this extension, and establish a more straightforward computational framework compared to existing techniques.

Method

This study employs a theoretical–analytical methodology built upon an extensive review of existing literature. Instead of relying on experimental procedures or computational simulations, the research is conducted through mathematical deduction, logical analysis, and theorem-driven formulation to establish a coherent generalization of determinant properties applicable to non-square matrices. The central aim is to develop an algebraically consistent framework that extends the classical notion of determinants from square matrices to rectangular ones, ensuring that their fundamental mathematical attributes remain preserved.

This research is conducted through library-based inquiry and theoretical examination (Merriam et al., 2015), synthesizing key definitions, propositions, and theorems drawn from classical linear algebra literature (Hill et al., 2008; Leon et al., 2010; Rorres et al., 2014). The adopted approach systematically revisits determinant theory to determine which fundamental attributes—such as linearity, antisymmetry, and multiplicative coherence—can be generalized to non-square matrices. Furthermore, the study investigates the structural consequences of extending the cofactor expansion principle, emphasizing its potential applicability beyond the conventional square matrix framework.

The methodological framework of this research is organized into four principal phases:

1. Formulation of Classical Determinant Theory:

A detailed review of determinant definitions applicable to square matrices is undertaken, focusing on traditional computational techniques such as Sarrus' rule, row-reduction procedures, and Laplace's cofactor expansion (DeFranza et al., 2009; Hill et al., 2008). This phase establishes the conceptual and theoretical basis necessary for the subsequent process of generalization.

2.Identification of Fundamental Determinant Properties:

This stage investigates essential determinant characteristics, including linearity with respect to rows, antisymmetry under row interchange, and invariance under matrix transposition (Leon et al., 2010; Poole et al., 2015). These features act as benchmark criteria for determining whether analogous properties persist when the determinant is extended to non-square matrices.

3. Construction of the Generalized Cofactor Expansion:

The classical cofactor expansion is reformulated to apply to rectangular matrices by redefining the notions of minors and cofactors to account for unequal numbers of rows and columns. The recursive cofactor principle is then adapted for cases where $m = n$, yielding a generalized determinant representation that aligns with the conceptual foundation of Radić while offering enhanced algebraic simplicity.

4. Validation through Theoretical Proof and Illustrative Examples:

The proposed generalization is substantiated through formal theorem derivations, corollaries, and worked examples. The derived results are systematically compared with established determinant properties of square matrices and earlier generalization models (Amiri et al., 2010; Abdollahi et al., 2015; Yurchenko et al., 2023) to verify both theoretical consistency and computational effectiveness.

The theoretical framework of this research is grounded in earlier studies on determinant generalization (Cullis et al., 1913; Radić et al., 1966; Joshi et al., 1980; Stanimirović et al., 1997). All definitions, propositions, and theorems are established through a deductive reasoning approach (Polya et al., 2004) to guarantee both logical soundness and internal coherence. The accompanying proofs aim to determine whether the fundamental axioms governing determinants of square matrices remain valid when extended to matrices of unequal dimensions.

In particular, the analysis examines whether the generalized determinant function adheres to the following axiomatic conditions:

1. $det(A) = 0$ if one row of A is consists entirely of zeros.
2. Exchanging two rows of A reverses the sign of the determinant.
3. Multiplying a single row by a scalar k results in the determinant being multiplied by k.
4. Adding a scalar multiple of one row to another does not alter the determinant's value.

The validation of the proposed generalized cofactor expansion is carried out by proving that the determinant expression formulated for rectangular matrices simplifies to the classical determinant when m=n. This verification upholds the consistency principle, ensuring alignment between the generalized formulation and the traditional definition. Furthermore, the study conducts an equivalence analysis comparing the proposed determinant with Radić's formulation, assessing whether both yield identical outcomes under specific dimensional conditions.

Accordingly, the overall methodological framework integrates comprehensive literature synthesis, theorem-driven deduction, and comparative theoretical analysis, thereby confirming both the validity and originality of the generalized determinant model developed in this research.

This session provides several definitions and theorems that will be used in the sub-discussion. Earlier than deriving the definition of the determinant of a matrix, consider first the that means of the submatrix stated in the following Definition 2.1.

Definition 2.1. (Selinger, 2018). If $A = (a_{ij})$ is an nxn matrix, then the submatrix of size $(n-1)x(n-1)$ obtained from A by deleting the i-th row and j-th column is known as the minor entry (i, j) of the matrix A and denoted by M_{ij} or $M_{ij}(A)$.

Definition 2.2. (Rorres, 2014; Gagliardi, 2009). Suppose matrix A is of size nxn, then the determinant of matrix A is expressed by det (A) and is defined as

$$\det(A) = \sum_{j=1}^{n} a_{ij}(-1)^{1+j} \det(M_{1j}) \quad (1)$$

Definition 2.3. (Leon, 2010; Gagliardi, 2009).The element cofactor (i, j) matrix $A = (a_{ij})$ is denoted by C_{ij} and defined as

$$C_{ij} = C_{ij}(A) = (-1)^{i+j} \det(M_{ij}(A)) \quad (2)$$

Note that $C_{ij}(A)$ does not depend on the elements of the i-th row or j-th column of matrix A. If equation (2) is substituted into equation (1), then the definition of the determinant can be rewritten as

$$\begin{aligned} \mathrm{Det}(A) &= a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n} \\ &= \sum_{j=1}^{n} a_{1j}C_{1j} \end{aligned} \quad (3)$$

So, det (A) is equal to the product of the first row elements of matrix A with their cofactors. Equation (3) is known as the cofactor expansion along the first row.

The properties of the determinants of a square matrix are stated in the following theorems:

Theorem 2.4. If $A = (a_{ij})$ is any square matrix and A^T is the transpose matrix of matrix A, then it applies $\det(A) = \det(A^T)$ (Hill, 2008)

Theorem 2.5. If matrix B is obtained from A by multiplying one row of matrix A by the constant $k \neq 0$, then $\det(B) = k det(A)$ (Hill, 2008).

Corollary 2.6. If A is a matrices of size nxn, and k is any scalar, then $\det(kA) = k^n \det(A)$

Theorem 2.7. If A and B are square matrices of the same size, then $\det(AB) = \det(A)\det(B)$.

Theorem 2.8. A square matrix A can be inverted (has an inverse) if $\det(A) \neq 0$.

Proof : If A is reversible then $AA^{-1} = I$. If we take the determinant, then

$$\det(AA^{-1}) = \det(I) = 1 \quad (4)$$

According to Theorem 2.7,

$$\det(AA^{-1}) = \det(A)\det(A^{-1}) \quad (5)$$

From (2.4) and (2.5), we get $(\det A)(\det A^{-1}) = 1$, therefore $\det A \neq 0$.

Result and Discussion

This section outlines the principal theoretical findings derived from generalizing determinant properties for non-square matrices through the cofactor expansion framework. Each presented theorem is supported by a rigorous proof and followed by a comparative discussion with existing results, highlighting the original contribution and theoretical significance of the present study.

The calculation of the determinant of a rectangular matrix is typically performed using the technique developed by Radic. In this paper, we generalize the calculation of these determinants from the cofactor expansion, which is commonly used for square matrices. The results obtained are presented in several theorems

Definition 3.1. If matrix $A = [a_{11} \quad a_{12} \quad a_{13} \quad \cdots \quad a_{1n}]$, then the determinant of matrix A is defined by

$$\det(A) = a_{11} - a_{12} + a_{13} - \cdots + (-1)^{1+n}a_{1n} = \sum_{i=1}^{n} a_{1i}$$

Note, if $B = [2 \quad 4 \quad 3 \quad 6]$, then $\det(B) = (2 - 4 + 3 - 6) = -5$.

We can generalize Definition 2.3 to non-square matrices, as stated in the following definition.

Definition 3.2 Let A be an $m \times n$ matrix where $m \neq n$. We define the *generalized determinant* of A recursively using cofactor expansion as

$$det(A) = \sum_{j=1}^{n} a_{1j}C_{1j}, \quad (6)$$

where $C_{1j} = (-1)^{1+j}det(M_{1j})$ and M_{1j} is the $(m-1) \times (n-1)$ minor matrix obtained by deleting the first row and j-th column of A.

If $m > n$, the determinant is computed using the transpose of A. This formulation guarantees reduction consistency, since for $m = n$ the definition reduces to the classical Laplace cofactor expansion (Leon, 2010; Hill, 2008).

The subsequent theorems show that the fundamental algebraic characteristics associated with square matrix determinants continue to hold true within the framework of this generalized definition.

Theorem 3.3. If A is a non-square matrix of size mxn where the entries in a particular row are zero, then $\det(A) = 0$.

Proof : By using cofactor expansion along the row containing zero entries, we get

$$det(A) = 0C_{i1} + 0C_{i2} + \cdots + (-1)^{i+n}0\, C_{in}.$$

This finding preserves the zero-row property known in square determinant theory (Hill, 2008)

Theorem 3.4. If A and B are mxn matrices with $m \leq n$, and matrix B is obtained from matrix A by swapping rows, then it holds $\det(A) = -\det(B)$.

Proof:

Let's say the matrix $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \end{bmatrix}$,

$B = \begin{bmatrix} a_{21} & a_{22} & \dots & a_{2n} \\ a_{11} & a_{12} & \dots & a_{1n} \end{bmatrix}$.

It is clear that matrix B is derived from matrix A by interchanging its rows. Note that

$$\begin{aligned}\det(B) = {} & a_{21}(a_{12} - a_{13} + \cdots + (-1)^n a_{1n} - a_{22}(a_{11} - a_{13} + \cdots + (-1)^n a_{1n}) \\ & + a_{23}(a_{11} - a_{12} + \cdots + (-1)^n a_{1n}) + \cdots + (-1)^{n+1}a_{2n}(a_{11} - a_{12} + \cdots + (-1)^n a_{1n-1}) \\ = {} & (a_{21}a_{12} - a_{21}a_{13} + \cdots + (-1)^n a_{21}a_{1n} - (a_{22}a_{11} - a_{22}a_{13} + \cdots + (-1)^n a_{22}a_{1n}) \\ & + (a_{23}a_{11} - a_{23}a_{12} + \cdots + (-1)^n a_{23}a_{1n}) + \cdots + (-1)^{n+1}(a_{2n}a_{11} - a_{2n}a_{12} + \cdots - \cdots \\ & + (-1)^n a_{1n-1}. \end{aligned} \quad (7)$$

Also,

$$\begin{aligned}\det(A) = {} & a_{11}(a_{22} - a_{23} + \cdots + (-1)^n a_{2n}) - a_{12}(a_{21} - a_{23} + \cdots + (-1)^n a_{2n}) \\ & + a_{13}(a_{21} - a_{22} + \cdots + (-1)^n a_{2n}) + \cdots + (-1)^{n+1}a_{1n}(a_{21} - a_{22} + \cdots + (-1)^n a_{2n}) \\ = {} & (a_{11}a_{22} - a_{11}a_{23} + \cdots + (-1)^n a_{11}a_{2n}) - (a_{12}a_{21} - a_{12}a_{23} + \cdots + (-1)^n a_{12}a_{2n}) \\ & + (a_{13}a_{21} - a_{13}a_{22} + \cdots + (-1)^n a_{13}a_{2n}) + \cdots + (-1)^{1+n}(a_{1n}a_{21} - a_{1n}a_{22} + \cdots - \cdots \\ & + (-1)^n a_{2n-1}) \end{aligned} \quad (8)$$

From (7) and (8), it can be seen that

$$\det(A) = -\det(B).$$

This confirms that the antisymmetric property remains valid for rectangular matrices, consistent with results in classical linear algebra (Leon, 2010; Radić, 1966).

Theorem 3.5. If two rows of A are identical, then det(A)=0.
Proof: Interchanging two identical rows changes the sign of the determinant (by Theorem 3.2) but does not change the matrix itself, hence det(A)=−det(A), leading to det(A)=0.
This property aligns with the Joshi determinant and remains consistent under the generalized expansion (Joshi, 1980).

Theorem 3.6. Let $A = (a_{ij})$ be an mxn matrix, if $m \leq n$, the determinant of A is

$$\text{Det}(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

Proof : Let B be the matrix acquired by moving the i^{th} row of A to the top, using $i-1$ interchange of adjacent rows. Thus $\det(B) = (-1)^{i-1}\det(A)$, but $b_{1j} = a_{ij}$ and $B_{1j} = A_{ij}$ for $j \in [1, n]$ and so,

$$det(B) = \begin{bmatrix} a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{(i-1)1} & \cdots & a_{(i-1)j} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & \cdots & a_{(i+1)j} & \cdots & a_{(i+1)n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Hence,

$$\begin{aligned} det(A) &= (-1)^{i-1} det(B) \\ &= (-1)^{i-1} \textstyle\sum_{j=1}^{n}(-1)^{1+j}\, b_{1j}\, det(B_{1j}) \\ &= (-1)^{i-1} \textstyle\sum_{j=1}^{n}(-1)^{1+j}\, a_{1j}\, det(A_{1j}) \\ &= \textstyle\sum_{j=1}^{n}(-1)^{1+j}\, a_{1j}\, det(A_{1j}). \end{aligned}$$

By applying the formula for cofactor along expansion the i^{th} row, obtained

$$\begin{aligned} \text{Det}(A) &= a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \\ &= \textstyle\sum_{j=1}^{n} a_{ij}C_{ij}. \end{aligned}$$

Example 3.6. Calculate the determinant of the matrix $A = \begin{bmatrix} 2 & 2 & 5 \\ 3 & 1 & 4 \end{bmatrix}$ with cofactor expansion
a). along the first row
b). along the second row.
Solution:

a). $\det(A) = \begin{vmatrix} 2 & 2 & 5 \\ 3 & 1 & 4 \end{vmatrix}$
$= 2|1 \quad 4| - 2|3 \quad 4| + 5|3 \quad 1|$
$= 2(-3) - 2(-1) + 5(2) = 6$

b). $\det(A) = \begin{vmatrix} 2 & 2 & 5 \\ 3 & 1 & 4 \end{vmatrix}$
$= -3|2 \quad 5| + 1|2 \quad 5| - 4|2 \quad 2|$
$= -3(-3) + 1(-3) - 4(0) = 6$

The consistency of these outcomes confirms that the *generalized determinant* remains invariant with respect to the choice of expansion position, analogous to the behavior observed in square matrix determinants (DeFranza et al., 2009; Leon et al., 2010). Moreover, this characteristic effectively overcomes the computational instability inherent in Radić's (1966) determinant formulation, which relies on complex multi-summation schemes instead of a recursive cofactor expansion process.

Corollary 3.7. If any two rows of the $m \leq n$ matrix are same, then the determinant value of the matrix is zero.
Proof : Suppose the determinant of a matrix A with size m×n with m≤n is |A|. Assume the two rows in matrix A are same. based on Theorem 3.4 by exchanging identical rows, we get -|A|. but the original matrix and the resulting matrix are the same, that is $|A| = -|A|$. Hence we get |A|=0.

Theorem 3.8. Suppose A and B are matrices with , and B is obtained from A by multiplying all entries in a certain row of matrix A with scalar k. So

$$\det(B) = k det(A).$$

Proof : If we expand along the i^{th} row of B to calculate its determinat, we get

$$\det(B) = b_{i1}B_{i1} + a_{i2}B_{i2} + \cdots + (-1)^{1+n}\, a_{in}B_{in}.$$

But the reason we have chosen the i^{th} row of B is that we know that $b_{ij} = ka_{ij}$ for $j = 1,2,\cdots,n$. Moreover, since the submatrices $B(i,j)$ will all have row i removed, and since this is the only place where B differs from A, we see that $A(i,j) = B(i,j)$. Thus, the cofactor B_{ij} for b_{ij} is the same as the cofactor $A_{ij}$for a_{ij}. So we have that

$$\begin{aligned} \det(B) &= ka_{i1}B_{i1} + ka_{i2}B_{i2} + \cdots + (-1)^{i+n}ka_{in}B_{in} \\ &= \text{k}(a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + (-1)^{i+n}a_{in}A_{in}) \\ &= k det(A). \end{aligned}$$

This verifies that the generalized determinant is linear with respect to rows, a property also established in square matrices (Rorres, 2014).

Corollary 3.9. Let's say that matrices A and B of size mxn with $m \leq n$ and B are acquared from A through multiplying all of the row entries of matrix A by the scalar $k \neq 0$. Then it happens

$$det(kA) = k^m\, det(A).$$

Proof : Since all m rows of A are multiplied by the scalar k to get kA, using the above theorem m times gives

$$\begin{aligned} \det(kA) &= (k)(k)(k)\cdots(k)\det(A) \\ &= k^m \det(A). \end{aligned}$$

Theorem 3.10. Suppose matrices A and B are mxn in size with $m \leq n$, and B is obtained from A by adding certain multiples of rows from A with other rows in A. Then it applies $\det(B) = \det(A)$.

Proof : Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{sn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, and assume that B is acquired by adding k times row s by row r, that is:

$$B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{sn} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{s1} + a_{r1} & ka_{s2} + a_{r2} & \cdots & ka_{sn} + a_{rn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

We calculate the determinant of B by using the expansion along the row r, then

$$det(B) = b_{r1}C_{r1} + \cdots + (-1)^{r+n} b_{rn}C_{rn}.$$

Note that $b_{rj} = ka_{sj} + a_{rj}$ for all $= 1, \cdots, n$. Therefore, the sub matrices $B(r,j)$ and $A(r,j)$ are the cofactors of A and B. This means that $B(r,j)$ comes from $A(r,j)$ which is obtained by adding k times the rows s with row . And because $B(r,j)$ and $A(r,j)$ are matrices $(m-1)x(n-1)$, we get

$$det\, B\,(r,j) = det\, A\,(r,j).$$

That is,

$$\begin{aligned} det(B) &= b_{r1}(-1)^{r+1} det\, B\,(r,1) + \cdots + b_{rn}(-1)^{r+n} det\, B\,(r,n) \\ &= (ka_{s1} + a_{r1})(-1)^{r+1} det\, A\,(r,1) + \cdots + (ka_{sn} + a_{rn})(-1)^{r+n} det\, A\,(r,n) \end{aligned}$$

$$\begin{aligned} det(B) &= ka_{s1}(-1)^{r+1} det\, A\,(r,1) + a_{r1}(-1)^{r+1} det\, A\,(r,1) + \cdots + ka_{sn}(-1)^{r+n} detA(r,n) + a_{rn}(-1)^{r+n} det\, A\,(r,n) \\ &= (ka_{s1}(-1)^{r+1} detA(r,1) + \cdots + ka_{sn}(-1)^{r+n} detA(r,n) + a_{r1}(-1)^{r+1} detA(r,1) + \cdots + a_{rn}(-1)^{r+n} detA(r,n) \end{aligned}$$

$$det(B) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{sn} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{s1} & ka_{s2} & \cdots & ka_{sn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{sn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

every matrix in one row is a multiple of the other row has a determinant of zero, so

$$det(B) = 0 + det(A) = det(A).$$

The generalized determinant formulation proposed in this study upholds all fundamental properties of the classical determinant while enabling a more efficient computation through a recursive expansion mechanism. In comparison with previous approaches (Radić et al., 1966; Joshi et al., 1980; Amiri et al., 2010; Abdollahi et al., 2015; Yurchenko et al., 2023), the present method:

1. Retains the complete set of classical determinant axioms,
2. Introduces a more straightforward and systematic computational procedure, and
3. Integrates the determinant concept within a unified framework applicable to both square and rectangular matrices.

The generalized cofactor expansion was shown to uphold fundamental properties: Zero row property: determinants vanish if a row is entirely zero. Row interchange antisymmetry: swapping rows changes the sign of the determinant. Row duplication: identical rows yield zero determinant.Linearity with respect to rows: multiplying a row by a scalar multiplies the determinant accordingly. Worked examples confirm consistency across different expansion positions, analogous to square matrices. This recursive formulation simplifies computation compared to Radić's multi-summation approach.

The findings align with Joshi et al., (1980), who emphasized algebraic coherence in determinant generalization, and Abdollahi et al. (2015), who proposed parallel algorithms for efficiency. However, unlike Radić's definition, the recursive cofactor expansion reduces computational complexity while retaining theoretical rigor. Yurchenko et al., (2023) highlighted symbolic computation compatibility, which is further supported by the recursive nature of this method. Thus, the proposed framework bridges the gap between square and rectangular determinants, offering both algebraic consistency and computational feasibility.

Conclusion

This study introduces a generalized determinant formulation for non-square matrices using recursive cofactor expansion. The approach preserves classical properties—linearity, antisymmetry, and reduction consistency—while simplifying computation compared to Radić and Joshi. The contribution lies in providing a unified theoretical framework applicable to both square and rectangular matrices, reinforcing linear algebra foundations and enabling broader applications in system analysis and symbolic computation.

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