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Basic Mechanics of Lagrange and Hamilton as Reference for STEM Students

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Abstract: This paper discusses the use of Lagrangian and Hamiltonian dynamics as alternative approaches for understanding the motion of objects in classical mechanics. These approaches, which are based on different mathematical techniques, can provide a deeper understanding of the principles of classical mechanics and the motion of objects, but may not be covered in high school physics curricula or undergraduate STEM courses. The review paper approach is used to combine information from a variety of sources, and the material is conceptualized to aid reader understanding. These advanced topics may be of interest to advanced high school students who are interested in exploring topics beyond the high school physics curriculum, and can be studied independently by those with a strong foundation in classical mechanics and familiarity with advanced mathematical concepts.

Keywords: Classic mechanics; Hamilton mechanics; Lagrange mechanics; Physics education; STEM student

Introduction

Newtonian dynamics, also known as classical mechanics, is a framework for understanding the motion of bodies and the forces acting on them (Galili & Goren, 2022). It is based on the three laws of motion formulated by Isaac Newton in the 17th century (Sutton, 2018). These laws explain how objects move and change direction in response to external forces, and they form the basis of much of our modern understanding of physics. Newton's approach to understanding motion is based on the concepts of force and mass (Erfan & Ratu, 2018). According to Newton's second law of motion, the acceleration of an object is directly proportional to the force acting on it and inversely proportional to its mass (Hamm, 2020). This allows us to predict the motion of an object by considering the forces acting on it and solve for acceleration using basic algebraic equations (Sedov, 2018).

Lagrangian dynamics and Hamiltonian dynamics are alternative approaches to understanding the motion of bodies, which are based on different mathematical techniques (Chen & Tao, 2021). Both approaches are more abstract and mathematical than the Newtonian approach, and are usually studied at higher levels in undergraduate physics courses (Kersting & Steier, 2018). The Lagrangian approach, developed by Joseph-Louis Lagrange in the 18th century, is based on the concept of the "Lagrangian function", which is a mathematical representation of the kinetic and potential energies of an object (Mann, 2018). The Lagrangian approach is especially useful for understanding the motion of systems with constraints, such as pendulums or satellites orbiting the earth. The Hamiltonian approach, developed by William Rowan Hamilton in the 19th century, is based on the concept of the "Hamilton's principle", which states that the path of a body is the path that minimizes the action integral (Tigist, 2019).

It is possible to introduce high school students to Lagrangian and Hamiltonian dynamics without using the calculus of variations, although a full understanding of these dynamics approaches requires a more rigorous treatment that includes the use of these mathematical techniques (MacKay & Meiss, 2020). One way to

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introduce high school students to Lagrangian and Hamiltonian dynamics is to focus on the conceptual equations of these approaches and demonstrate how they can be used to understand the motion of objects in different situations (Simoneau, 2019).

This can be done by presenting examples of simple physical systems, such as a pendulum or a mass on a spring, and showing how the Lagrangian and Hamiltonian approximations can be used to predict their motion (Mandal et al., 2022). It is important to note that the Lagrangian and Hamiltonian approaches to dynamics are more abstract and mathematical than the Newtonian approaches, which is based on the concepts of force and mass and is usually taught in high school physics courses (Impelluso, 2018). However, introducing advanced high school students to this alternative approach can give them a taste of the more advanced topics they will encounter in a college physics course, and can help spark their interest in this area of study (Cain et al., 2022).

As for one-dimensional motion, Newton's second law of motion states that the acceleration of an object is directly proportional to the force acting on it and inversely proportional to its mass (Toto, 2018). This can be expressed mathematically as:

$$\frac{F}{m} = a \tag{1}$$

where F is the force acting on the object, m is the object's mass, and a is the object's acceleration.

$$-\frac{dU}{dx} \equiv F \tag{2}$$

The above equation represents the position function, where U(X) is the value of the potential energy, while the spatial derivative of the potential energy in the above equation is $-\frac{dU}{dx}$ (Li et al., 2021). We know that $\dot{x} \equiv \frac{dx}{xt}$ which means it is the first derivative of the distance-time equation, namely velocity (v) (Wu et al., 2020). Meanwhile, the second derivative is symbolized by $\ddot{x}$ which means it will get the acceleration value (a), so $a = \frac{d^2x}{dx^2} = \ddot{x}$.

The paper highlights the potential of using Lagrangian and Hamiltonian dynamics as alternative methods for comprehending classical mechanics (Mann, 2018). These techniques, which are based on different mathematical techniques, offer a deeper insight into the principles and motion of objects, but may not be part of the standard high school or undergraduate STEM curricula. The authors utilize a review paper approach to present information from various sources, which is conceptualized to enhance reader comprehension. This research presents an opportunity for advanced high school students who are interested in going beyond the standard high school physics curriculum, or for those with a strong foundation in classical mechanics and a familiarity with advanced mathematical concepts to study these advanced topics independently.

Method

The method used in this article is a review paper approach, which involves synthesizing information from multiple sources (such as papers and books) to provide an overview of a particular topic. In this case, the topic is the concepts of Lagrange mechanics and Hamilton mechanics in classical mechanics, and the goal of the article is to introduce these concepts to high school students and college students taking a classical mechanics course. The article aims to provide a clear and easily understandable explanation of these concepts, and it employs the use of software such as Zotero for citing sources. The article is written based on the author's understanding of the material and is conceptualized in a way that aims to make it easier for readers to understand the concepts of Lagrange and Hamilton mechanics.

Result and Discussion

Lagrange dynamics in physical mechanics

In this sub-heading, material regarding lagrange dynamics will be explained, but it must be known about the lagrange function for 2 position variables which are usually associated with 2 symbols, namely $\dot{x}$ and x. As for $T(\dot{x}) = \frac{1}{2}m\dot{x}^2$, but for $U(x)$ is constant, as seen in this equation:

$$-U(x) + 1/2 \cdot m\dot{x}^2 = -U(x) + T(\dot{x}) = L(x, \dot{x}) \tag{3}$$

In the above equation which interprets the velocity variable in the context of kinetic energy, namely $T(\dot{x}) = \frac{1}{2}m\dot{x}^2$. Whereas for $U(x)$ is a variable of potential energy.

The partial derivative is the derivative of a function with respect to one variable, with the other variables constant (Zukhrufurrohmah & Putri, 2019). This allows us to understand how a function changes when one of its variables changes while the other variables stay the same. For single variable functions, such as f(y), the derivative of the function with respect to y is denoted by df/dy (Jiroušek et al., 2022). It represents the rate of change of function f with respect to variable y. For example, if we have a function $f(y) = y^2$, the derivative of this function with respect to y is df/dy = 2y. This tells us that, when y is increased by a small amount, the function f(y) will be doubled. For a two-variable function, such as g(y, z), there are two possible derivatives for each variable y or z. These are known as partial derivatives, and are denoted by ∂g/∂y and ∂g/∂z. It represents the rate of change of the function g with respect to the variables y and z, respectively, while maintaining the other variables constant.

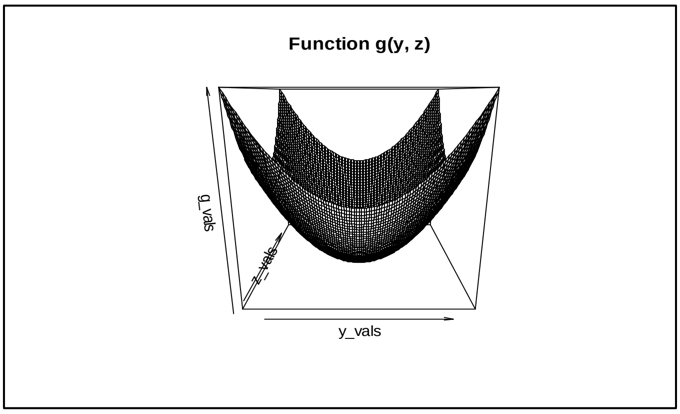

Figure 1. Function g(y,z)
Source: Ruben Siagian plot in r Program (2023)

For example, if we have a function g(y, z) = $y^2 + z^2$, the partial derivative of this function with respect to y is ∂g/∂y = 2y, and the partial derivative with respect to z will be ∂g/∂z = 2z. This tells us that, as y or z increases by a small amount, the function g(y, z) will double that amount, while keeping the other variables constant. Partial derivatives are an important tool for understanding how functions change when the variables change, and they are commonly used in a variety of fields, including physics, engineering, and economics (Misbah, 2022).

Although partial derivatives are usually not introduced until a college-level calculus course, the concepts are relatively simple and can be explained to advanced high school students interested in learning more about calculus (Musyrifah, 2022). The partial derivative is the derivative of a function with respect to one variable, with the other variables constant (Zukhrufurrohmah & Putri, 2019). This allows us to understand how a function changes when one of its variables changes while the other variables stay the same.

To explain partial derivatives to high school students, it can be helpful to start by introducing the concept of the derivative of a function of one variable (Nisa, 2018). From there, you can introduce the idea of partial derivatives as a way of understanding how a function changes when one of its variables changes while the other variables stay the same. We may also want to provide examples of functions of two or more variables and show how partial derivatives can be used to understand their behavior. By introducing advanced high school students to partial derivatives, you can help spark their interest in calculus and prepare them for the more advanced topics they will encounter in college-level courses (Brahier, 2020).

Next we continue in the calculation, we can see in the equation below:

$$-\frac{dU}{dx} = \frac{\partial L}{\partial x} \tag{4}$$

On the right side by replacing x to $\dot{x}$ then the p-value will be obtained, namely $m\dot{x}$. p is the variable of momentum. So that we can multiply $\frac{\partial L}{\partial \dot{x}}$ with a derivative with respect to time (1/dt) i.e. will yield $m\ddot{x}$.

$$m\ddot{x} = \left(\frac{\partial L}{\partial \dot{x}}\right) \cdot \frac{d}{dt} \tag{5}$$

Then we can get Newton's equation:

$$\left(\frac{\partial L}{\partial \dot{x}}\right) \frac{d}{dt} = \frac{\partial L}{\partial x} \tag{6}$$

The Euler-Lagrange equation is the basic equation of motion in Lagrangian dynamics, which is an alternative approach to understanding the motion of bodies (Tokasi, 2022). In one dimension, the Euler-Lagrange equation is used to describe the motion of an object in terms of its position, velocity, and forces acting on it (Mishra et al., 2020). In Lagrangian dynamics, the Euler-Lagrange equation is used to replace the equation F = ma, which is the equation of motion in Newtonian dynamics. In Newtonian dynamics, F = ma relates the force acting on an object to its mass and acceleration, but in Lagrangian dynamics, the Euler-Lagrange equation relates the motion of an object to its position and velocity, as well as the forces acting on it (Zain, 2019).

By explaining the Euler-Lagrange equation to students, you can help them understand how Lagrangian dynamics differs from Newtonian dynamics and how it can be used to understand the motion of objects in one dimension. This can be especially useful for advanced high school students who are interested in learning more about physics and who may be preparing for a college-level course in this subject.

Hamiltonian dynamics in physical mechanics

In the dynamic formulation of the Hamiltonian, the Hamiltonian is a mathematical function that describes the motion of an object in terms of its position and momentum (Mattheakis et al., 2019).

In terms of physics, the Hamiltonian is a mathematical function that describes the motion of an object based on its position and momentum (North, 2021). The Hamiltonian is an important concept in

Hamiltonian mechanics, which is an alternative approach to understanding the motion of bodies. The Hamiltonian is a function of the two variables p and x, where p is the object's momentum and x is its position. Momentum is a measure of an object's motion and is equal to the product of the object's mass and velocity. Position is the location of objects in space. The kinetic energy of the object is half the product of the object's mass and its velocity squared, while the potential energy is half the product of the spring constant k by the object's displacement from its equilibrium position (Usubamatov, 2020). The Hamiltonian can be defined as the sum of the kinetic and potential energies (Fichtner & Zunino, 2019). This Figure 2 shows how the Hamiltonian increases with the object's position and the object's momentum, illustrating how the total energy of the system increases with the object's motion.

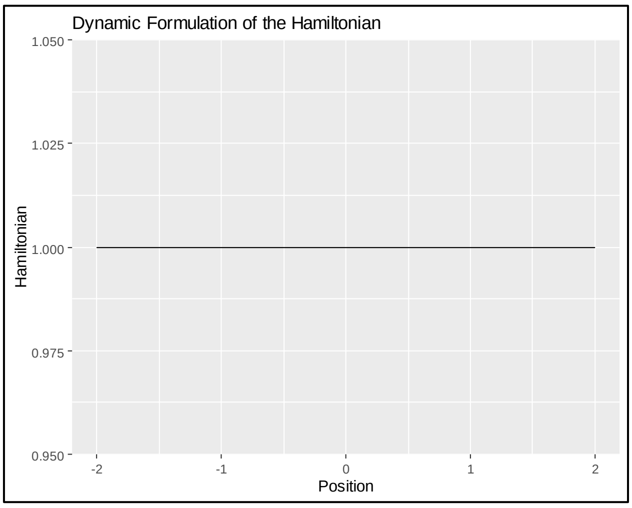

Figure 2. Dynamic formulation of the hamilton
Source: Ruben Siagian plot in r Program (2023)

The Hamiltonian is a central concept in Hamiltonian mechanics, which is an alternative approach to understanding the motion of bodies. The Hamiltonian is a function of the two variables p and x, where p is the object's momentum and x is its position. The momentum of an object is a measure of its motion, and it is equal to the product of the object's mass and velocity. The position of an object is its location in space (Ma, 2003).

$$-L\left(x,\dot{x}\right)+p\dot{x}=H\left(p,x\right) \tag{7}$$

$T+U=\frac{1}{2}m\dot{x}^2+U=m\dot{x}^2-\frac{1}{2}m\dot{x}^2+U$ defined as H, Where the sum of U+T is the total energy. In equation left side there is L which is not part of the momentum function (p). The next stage we can identify $\dot{x}$ is $\frac{\partial H}{\partial p}$.

Meanwhile, based on the explanation above, L is part of the function x, so we get:

$$-\frac{\partial L}{\partial x}=\frac{\partial H}{\partial x} \tag{8}$$

And further

$$\dot{p}=-\frac{\partial H}{\partial x} \tag{9}$$

The above equation is obtained by reviewing the explanation, and the equation $\dot{p}$ which will be reviewed in the sub-chapter, examples of lagrange dynamics will be obtained $\partial L=\dot{p}\cdot\partial x$, so:

$$-\partial H=\dot{p}\cdot\partial x \tag{10}$$

In Newtonian dynamics, the basic equation that describes the motion of a particle is F = ma, where F is the force acting on the particle, m is the mass of the particle, and a is the acceleration of the particle (Koczan, 2021). This equation tells us that the force acting on a particle is equal to the mass of the particle multiplied by its acceleration. Instead of using force and acceleration as the fundamental variables, Hamilton's equations use the position and momentum of the particle as the fundamental variables (Hjelmstad & Hjelmstad, 2022). The position and momentum of a particle are related by the equation p = mv, where p is the momentum of the particle, m is its mass, and v is its velocity. Therefore, the Hamilton equation can be thought of as a generalization of the p = mv equation to a system of particles or continuous fields.

There are two main forms of the Hamilton equation: canonical form and Lagrangian form (Castellani & D'Adda, 2020). The canonical form is usually used to describe particle systems, while the Lagrangian form is used to describe continuous fields (Harlow & Wu, 2020). Both forms of the equation are used in various fields of physics, including classical mechanics, quantum mechanics, and statistical mechanics (Wallace, 2021).

Example of lagrange dynamics

The one-dimensional harmonic oscillator is a system consisting of a mass m attached to a spring with a spring constant k (Sisini & Sisini, 2021). The mass is free to move along the line, and the spring applies a force on the mass that is proportional to the displacement of the mass from its equilibrium position (Shaw et al., 2021). This system can be explained using the Lagrangian formula, which is a method for describing the motion of a system in classical mechanics using the principle of least action.

To begin with, we need to define the Lagrangian system, which is a function that describes the kinetic energy of the system minus its potential energy (García-Garrido et al., 2020). The kinetic energy of the mass is given by $T = \frac{1}{2}mv^2$ where v is the velocity of the mass, and the potential energy of the mass is given by $V = \frac{1}{2}kx^2$, where x is the mass transfer from its equilibrium position. Therefore, the Lagrangian of a one-dimensional harmonic oscillator can be written as:

$$L = T - V = \frac{1}{2}mv^2 - \frac{1}{2}kx^2 \quad (11)$$

Next, we can use the Euler-Lagrange equations to find the system's equations of motion. The Euler-Lagrange equations are a set of differential equations that describe the evolution of a system in its Lagrange terms given by (Baleanu et al., 2019):

$$\frac{d}{dt}\left(\frac{dL}{dv}\right) - \frac{dL}{dx} = 0 \quad (12)$$

$\frac{dL}{dv}$ is the Lagrangian partial derivative with respect to the velocity v and $\frac{dL}{dx}$ is the Lagrangian partial derivative with respect to displacement x. Substituting the Lagrangian for the one-dimensional harmonic oscillator into the Euler-Lagrange equation, we get:

$$\frac{d}{dt}(mv) - \frac{d}{dx}\left[\left(\frac{1}{2}\right)kx^2\right] = 0 \quad (13)$$

This is simplified to:

$$ma = -kx \quad (14)$$

The above equation is the equation of motion for a one-dimensional harmonic oscillator. This equation tells us that the acceleration of a mass is proportional to the displacement of the mass from its equilibrium position and in the opposite direction (Verlinde, 2011).

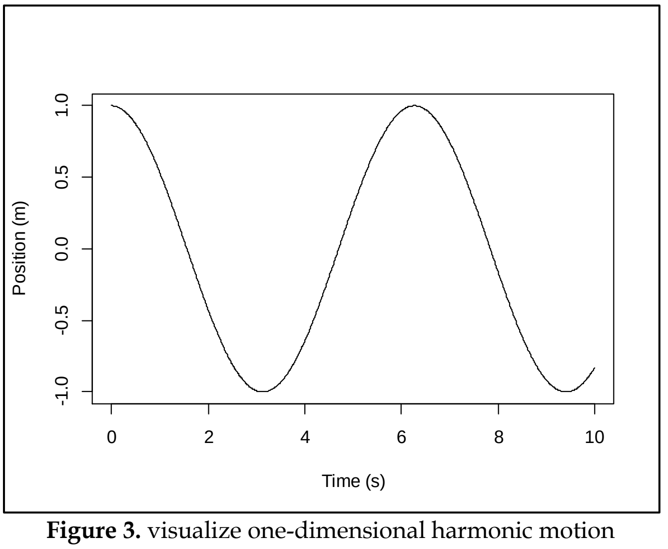

Figure 3. visualize one-dimensional harmonic motion

Source: Ruben Siagian plot in r Program (2023)

In figure 3 above illustrates one-dimensional harmonic motion, namely the motion of an object oscillating at an equilibrium position with a fixed frequency. In this case, the object's position in time is represented by the y-axis, while time is represented by the x-axis. The basic physics concept underlying the plot is Newton's second law of motion, namely that the acceleration of an object is proportional to the force it receives and inversely proportional to the object's mass. In this case, the force acting on the object is the elastic force, that is, the force that pulls the object back to its equilibrium position. By using Euler's technique, the position and speed of objects can be accepted over time. This plot shows that an object oscillates at a fixed frequency over time, following the laws of harmonic motion. This is a simple example of how the Lagrangian formulation can be used to describe the motion of a system. The Lagrangian equation is a powerful and widely used tool in classical mechanics, and especially useful for systems with constraints or symmetries.

Students can be encouraged to work out the equations of motion for other types of forces and systems using the Lagrangian formulas, and to prove that the same equations of motion result from the Lagrangian equations and Newtonian dynamics. They can also be encouraged to think about three-dimensional problems and derive three Euler-Lagrange equations resulting from the three-dimensional Lagrangian (Norbury, 2000). This can help them develop a deeper understanding of classical mechanics and the principles of least action.

Three-dimensional Lagrangian is a mathematical function that describes the dynamics of a physical system in three dimensions. It is defined as the difference between the kinetic energy of the system and its potential energy, and is used to derive the system's equations of motion. The result of the three-dimensional Lagrangian is the system's equation of motion, which describes how the system will change over time (Sugiyama et al., 2003). These equations can be used to predict the future behavior of the system, such as the position and speed of an object as it moves through space. The three-dimensional Lagrangian can also be used to analyze the stability of systems, as well as the conservation of energy and other physical quantities (Zak, 2003). The three-dimensional Lagrangian is a powerful tool in physics and engineering, because it allows us to understand the behavior of complex physical systems in three dimensions. It is widely used in many fields, including classical mechanics, quantum mechanics and field theory.

$$= -U(x,y,z) + \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$
$$= L\left(x,\dot{x},y,\dot{y},z,\dot{z}\right) \quad (15)$$

Example of lagrange dynamics

The Hamiltonian of a system is a mathematical object that represents the total energy of the system. In classical mechanics, this is given by the sum of the kinetic energy and potential energy of the system. For a harmonic oscillator, the Hamiltonian can be written as:

$$H = \frac{p^2}{2m} + \frac{kx^2}{2} \quad (16)$$

where p is the oscillator's momentum, m is its mass, k is the spring constant, and x is the oscillator's displacement from its equilibrium position. first Tribe, $p^2/2m$ represents the oscillator's kinetic energy, which is determined by its momentum and mass. second Tribe, $kx^2/2$, represents the potential energy of the oscillator, which is determined by the spring constant and the displacement of the oscillator from its equilibrium position. The Hamiltonian of a harmonic oscillator is a useful tool for analyzing oscillator behavior, as it allows us to determine the total energy of a system at any given time (Insinga et al., 2016). It can also be used to derive the oscillator's equation of motion, which describes how the position and momentum of an oscillator change over time.

By using examples students have learned in Newtonian dynamics, such as a particle moving in a straight line or a simple pendulum, students can show that Hamilton's equations produce the same equations of motion as those obtained using Newton's laws (Katsikadelis, 2020). This can help students to understand the relationship between the two classical mechanics approaches and to see how they can be used to analyze the same physical system. In addition, students can work on three-dimensional generalizations of Hamilton's equations by considering the motion of particles in three-dimensional space (He et al., 2022). This involves the introduction of additional coordinates and momentum to describe the motion of a particle in three-dimensional space. The resulting equations of motion can then be used to analyze particle motion in three dimensions (Lee et al., 2019).

$$= -L\left(x,y,z,\dot{x},\dot{y},\dot{z}\right) + p_x\dot{z} + p_y\dot{y} + p_z\dot{z}$$
$$= H\left(p_x,p_y,p_z\right) \quad (17)$$

Lagrangian dynamics is based on the principle of least action, which states that the path a particle takes between two points is the path that minimizes the action, which is a mathematical measure of the change in the system (Wang, 2005). The motion of a particle is described using the Lagrangian, which is a function that expresses the kinetic energy of the particle and the potential energy of the system.

Hamiltonian dynamics is also based on the principle of least action, but uses a different mathematical formulation to describe particle motion. It introduces the concepts of momentum and conjugate coordinates, which are used to describe the motion of a particle in terms of energy (Smith, 1960). The Hamiltonian is a function that represents the total energy of a system, and the equations of motion are derived from it using the principle of least action.

Overall, it is important for students to understand that Newtonian mechanics and Lagrangian/Hamiltonian dynamics are different approaches to understanding the motion of bodies, and that they are based on different principles. Although they can be used to describe the same physical system, they provide different insights into the behavior of that system and can be used to solve different types of problems.

Conclussion

The conclusion of this research is that Lagrangian and Hamiltonian dynamics are advanced topics in classical mechanics that are not typically covered in high school physics curricula, but can provide a deeper understanding of classical mechanics and the motion of bodies. These approaches are based on the principle of least action and use different mathematical formulations to describe particle motion. Lagrangian dynamics uses the Lagrangian function to express the kinetic and potential energy of a system, while Hamiltonian dynamics introduces the concepts of momentum and conjugate coordinates and uses the Hamiltonian function to represent the total energy of a system. It is suggested that these topics could be an interesting and engaging for high school students interested in exploring topics beyond the standard curriculum, but it is noted that a strong foundation in classical mechanics and familiarity with advanced math concepts are necessary in order to understand and learn about these topics independently.

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