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Legendre Polynomials are Associated in the Schrodinger Polar Equation of Helium Ions

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Abstract: The helium ion is one of the ions classified as a hydrogenic atom because it has only one electron in its outer orbital so that the helium ion problem can be solved using the Schrodinger equation approach. The Schrodinger equation of the polar part of the helium ion can be solved by the associated legendary polynomial function, $P_l(v) = \frac{1}{2^l l!}\frac{d^l}{dv^l}(v^2-1)^l$. The polar function expresses the angle formed by the vector of the electron's position relative to the z-axis of the helium ion's nuclear position. And it is produced that the polar function of the hydrogenic atom has the same polar function both in position space and in momentum space $({}^4_2He^+)$.

Keywords: Helium ion; Legendre polynomial; Schrodinger equation

Introduction

The photoelectric effect experiment and the Compton effect cannot be well explained by the theories of classical physics, but can be explained using the theories of quantum physics, as stated in the research of Zeng (2021) and Ferent (2019). In the experiment, particles and waves cannot be distinguished when to behave as particles and when to behave as waves, because these two things have a very fundamental relationship (Hari, 2019). Zeng (2021) proposes a modified quantum concept to bridge the gap between classical and quantum physics, providing a more comprehensive explanation for the phenomenon. However, Katzir (2006) points out that Richardson's theory of thermodynamics of the photoelectric effect, although not widely accepted, also offers a potential explanation. Klassen (2011) emphasizes the importance of accurately describing the history of this phenomenon, particularly in the context of quantum physics. In quantum physics, the behavior of microscopic objects is discussed, massive or non-massive objects that are about the diameter of an atom or smaller and have a dual behavior (Halim & Herliana, 2020). In addition, in quantum physics also discusses some polynomial functions (Supriadi et al., 2022). The dualism of particle waves has been fundamental in the development of new quantum theories, for example being the background for the formation of the Schrödinger equation as a fundamental equation in quantum mechanics (Chowdhury et al., 2021). Particle wave dualism being one of the most interesting concepts in quantum mechanics, the study of wave and particle interpretations is very different from the views of classical physics (Li et al., 2023).

A hydrogenic atom is an atom that retains only one electron. Simply put, this atom has only one electron, similar to the Hydrogen atom (Pratikha et al., 2022). The

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hydrogen atom is a very simple atom and has a light mass because it consists of one proton and one electron in its orbital (Supriadi et al., 2020). Hydrogen has three isotopic forms consisting of protium, deuterium and tritium (Urrestizala et al., 2023). Protium is an isotope of hydrogen that has 1 electron and 1 proton. Deuterium is an isotope of hydrogen that has 1 electron, 1 proton and 1 neutron. While tritium has 1 electron, 1 proton and 2 neutrons (Karomah et al, 2021).

Helium is one of the noble gas atoms that has 2 electrons in its orbitals, 2 protons, and 2 neutrons. If one of the electrons in helium ionizes, it becomes a helium ion characterized as a hydrogenic atom (Widiastuti, 2019). Gautreau et al. (2006) state that a hydrogenic atom is a single-electron atom in its outermost orbital. Thus, the helium ion ($^{4}_{2}He^{+}$) behaves the same as hydrogen, the difference being that its nucleus is positively charged 2e, where Z = 2 is the atomic number.

In everyday life, helium ions are widely used as fillers for hot air balloons because they have low vapor density and viscosity and stable concentrations, as coolants, and as a substitute for normal air to help underwater diving on the seabed (Nurroniah et al., 2023). In addition, in the liquid state helium serves as a persistent coolant and is produced for its application as a semiconductor magnet, particularly in techniques such as Magnetic Resonance Imaging (MRI) and Nuclear Magnet Resonance (NMR) (Rillo et al., 2015). Alimah (2016) mentioned that helium can function as a refrigerant because it is classified as an inert gas or ideal gas, making it useful for heat transfer applications. Saputro (2019) said that helium ions can be used as a cold atmosphere plasma-based sterilizer (CAP) which effectively kills microbes in mangosteen fruit. According to Berganza et al. (2013), helium gas can also be used as an adjunct therapy for respiratory diseases and its potential in protecting the myocardium from ischemia.

Schrödinger's equation is an equation used to explain the nature or state of a particle through the principles of the law of conservation of energy and the de Broglie hypothesis (Shukron et al., 2022). The Schrödinger equation must also behave well, that is, it is linear and homogeneous (Sudiarta, 2019).

In general, the Schrodinger equation in spherical coordinates is written as (Kadri & Sani, 2017).

$$\frac{h^2}{m_0}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\Psi(\mathrm{r},\theta,\varphi)-\frac{Ze^2}{4\pi\varepsilon_0 r}\Psi(\mathrm{r},\theta,\varphi)=E\Psi(\mathrm{r},\theta,\varphi) \quad (1)$$

To solve the equation, the following variable separation method can be used by introducing two variable functions.

$$\psi_{(r,\theta,\phi)}=R(r)Y(\theta,\phi) \quad (2)$$

So, the equation is obtained:

$$\left[\frac{1}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right)+\frac{2\mu r^2}{\hbar^2}[E-V(r)]\right]=-\left[\frac{1}{Y(\theta,\phi)sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial Y(\theta,\phi)}{\partial\theta}\right)+\frac{1}{Y(\theta,\phi)sin^2\theta}\frac{\partial^2 Y(\theta,\phi)}{\partial\phi^2}\right] \quad (3)$$

If the radial equation is assumed to be a constant, then the radial and angulatory equations become:λ

a. $$\left[\frac{1}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right)+\frac{2\mu r^2}{\hbar^2}[E-V(r)]\right]=\lambda \quad (4)$$

b. $$\left[\frac{1}{Y(\theta,\phi)sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial Y(\theta,\phi)}{\partial\theta}\right)+\frac{1}{Y(\theta,\phi)sin^2\theta}\frac{\partial^2 Y(\theta,\phi)}{\partial\phi^2}\right]=-\lambda \quad (5)$$

(Supriadi, et al., 2019).

The angulatory section of the Schrödinger equation is a partial differential equation, the solution of which can be done using the variable separation method $Y_{(\theta,\phi)}=\Theta(\theta)\Phi(\phi)\theta\phi m_l^2$. So that the equation of the polar part that depends on the angle and the equation of azimuth that depends on the angle, using the constant then the equation is obtained as follows:

a. Polar equation depends θ

$$\left[\frac{sin\theta}{\Theta(\theta)}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}\right)+\lambda sin^2\theta\right]=m_l^2 \quad (6)$$

b. The azimuth equation depends ϕ

$$-\left[\frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}\right]=m_l^2 \quad (7)$$

(Supriadi, et al., 2019).

The polar part equation as shown in equation (6) can be solved using the polynomial method.

Polynomials are one of the infinitely power series functions that are often used in the solution of differential equations. Hidayatullah et al. (2017) states that a polynomial is a multiterm of degree n, with n numerical numbers. A polynomial in one variable with a constant coefficient has the following general form:

$$a_n X^n + a_{n-1}X^{n-1}+\cdots+a_2X^2+a_1X^1+a_0 \quad (8)$$

With, named as coefficients, named as variables of the power , n, n − 1, ..., 2, 1 are called ranks, and are called fixed terms. Discussion of ordinary differential polynomials (ODE), which includes hypergeometric equations. In general, hypergeometric equations can be written as follows: $a_n \neq 0a_n,\ a_{n-1},\dots,a_2,\ a_1X^n$, $X^{n-1},\dots,\ X^2,\ Xa_0$

$$s(x)F^{"}(x) + t(x)F'(x) + \lambda F(x) = 0 \quad (9)$$

(Aboites & Ramírez, 2019).

Polynomial Legendre was first introduced in 1782 by Adrien-Marie Legendre. Spherical harmonic solutions constitute one of the important classes of special functions very closely related to Legendre polynomials (Nasution et al., 2023). The properties of the Legendre polynomial are if v = 1, then $P_l(v) = 1$; If v = -1, then $P_l(-1) = (-1)^l$; and if v = -v, then $P_l(-1) = (-1)^l P_l(v)$. Legendre polynomial functions are widely used in differential equations. Jena et al. (2020) states that the Legendre equation can be used to solve fractional order differential equations. The research is in harmony with research by Moshtaghi et al. (2021) those who use the Legendre equation as a numerical solution of a distributed order fractional differential equation.

Associated legendre polynomials are obtained by derivation from angular equations. With is an associated legendre function, defined by: $P_l^m(cos\theta)$

$$P_l^m = (1-x^2)^{\frac{|m|}{2}}(\frac{d}{dx})^{|m|}P_l(x) \quad (10)$$

And is the 1th legendary polynomial$P_l(x)l$, defined by Rodriges' formula, which is:

$$P_l(x) = \frac{1}{2^l l!}(\frac{d}{dx})^l(x^2-1)^l \quad (11)$$

(Abdelhakem & Moussa, 2023 ; Ciftja, 2022).
The solution to the polar legendre equation is written as:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[\lambda - \frac{m^2}{\sin^2\theta}\right]\Theta = 0 \quad (12)$$

Based on this discussion, this study aims to apply associated legendre polynomials in the polar schrodinger equation of helium ions. The results of the study are expected to add references to the application of the method of associated legendre polynomials in the polar schrodinger equation on helium ions.

Method

This research is a type of non-experimental research. The method used is study literature, to develop existing theories by solving the Schrodinger equation of the polar part of the helium ion using the Legendre equation,

$$(1-v^2)\frac{d^2\Theta}{dv^2} - 2v\frac{d\Theta}{dv} + \lambda\Theta = 0 \quad (13)$$

The solution can be written in the form of infinite power series or polynomial functions, namely:

$$\Theta(v) = \sum_{n=0}^{\infty} a_n v^{k+n} \quad (a_0 \neq 0) \quad (14)$$

The first and second derivatives of the polynomial functions above are:

$$\frac{d\Theta}{dv} = (k+n)\sum_{n=0}^{\infty} a_n v^{k-1+n} \quad (15)$$

and $\frac{d^2\Theta}{dv^2} = (k+n)(k-1+n)\sum_{n=0}^{\infty} a_n v^{k-2+n}$ (16)

By substituting equations (14), (15) and (16) to equation (13) it is obtained:

$$k(k-1)a_0v^{k-2} + k(k+1)a_1v^{k-1} + \sum_{n=0}^{\infty}[(k+n+2)(k+n+1)a_{n+2} - (k+n)(k+n-1)a_n - 2(k+n)a_n + \lambda a_n]\, v^{k+n} = 0 \quad (17)$$

Equation (17) requires that each of its term coefficients be zero for all values, so that $v\, k = 0$ it is a_1 of such an arbitrary value and $k = 1$ and the recursion equation:

$$a_{n+2} = \frac{[(k+n)(k+n-1)+2(k+n)-\lambda]}{(k+n+2)(k+n+1)}a_n \quad (18)$$

Polynomial function, $\Theta(v)$ having a convergence interval $-1 < v < 1$. If , the interval of convergence becomes by $v = \cos\theta - 1 \le v \le 1$replacing with an integer always positive i.e. . $\lambda l(l+1)$. So that equation (18) can be expressed as:

$$a_{l-2r} = (-1)^r\left[\frac{(l!)^2(2l-2r)!}{(2l)!r!(l-2r)!(l-r)!}\right]a_l \quad (19)$$

And the solution of the Legendre equation (14) can be expressed as:

$$\Theta_l(v) = P_l(v) + Q_l(v) = \sum_{r=0}^{N}\frac{(-1)^r(2l-2r)!v^{l-2r}}{2^l r!(l-r)!(l-2r)!} \quad (20)$$

Where $P_l(v)$ is a first-order Legendre polynomial and is a second-order (divergent) Legendre polynomial. $Q_l(v)$TheRodrigues umus for the Legendre polynomial is

$$P_l(v) = \frac{1}{2^l l!}\frac{d^l}{dv^l}(v^2-1)^l \quad (21)$$

Obtained Legendre polynomials for the range l as presented in table 1.

Table 1. Legendre Polynomial $\boldsymbol{P_l(v)}$

l	$\boldsymbol{P_l(v)}$
0	1
1	v
2	$\frac{1}{2}(3v^2-1)$
3	$\frac{1}{2}(5v^3-3v)$

The associated Legendre equation is formulated:

$$(1-\nu^2)\frac{d^2\Theta}{d\nu^2}-2\nu\frac{d\Theta}{d\nu}+\left\{l(l+1)-\frac{m^2}{1-\nu^2}\right\}\Theta=0 \quad (22)$$

by being solved by $m^2 < l^2$introducing in series notation (solutions related to the Legendre Polynomial), namely:Θ

$$\Theta = (1-\nu^2)^{\frac{m}{2}}u(\nu) \quad (23)$$

Then the first and second differentials of equation (23) are obtained:

$$\frac{d\Theta}{d\nu}=\frac{m}{2}(1-\nu^2)^{\frac{m}{2}-1}(-2\nu)\,u(\nu)+(1-\nu^2)^{\frac{m}{2}}\frac{du}{d\nu}$$
$$= (1-\nu^2)^{\frac{m}{2}}\frac{du}{d\nu}-m\nu u(1-\nu^2)^{\frac{m}{2}-1} \quad (24)$$

And

$$\frac{d^2\Theta}{d\nu^2}=\frac{m}{2}\left(\frac{m}{2}-1\right)(1-\nu^2)^{\frac{m}{2}-2}(4\nu^2)\,u(\nu)-\frac{m}{2}(1-\nu^2)^{\frac{m}{2}-1}(-2)\,u(\nu)+\frac{m}{2}(1-\nu^2)^{\frac{m}{2}-1}(-2\nu)\,\frac{du}{d\nu}+\frac{m}{2}(1-\nu^2)^{\frac{m}{2}-1}(-2\nu)\frac{du}{d\nu}+(1-\nu^2)^{\frac{m}{2}}\frac{d^2u}{d\nu^2} \quad (25)$$

or

$$\frac{d^2\Theta}{d\nu^2}=(1-\nu^2)^{\frac{m}{2}}\frac{d^2\Theta}{d\nu^2}-2m\nu(1-\nu^2)^{\frac{m}{2}-1}\frac{du}{d\nu}-mu(1-\nu^2)^{\frac{m}{2}-1}+2m\nu^2u\left(\frac{m}{2}-1\right)(1-\nu^2)^{\frac{m}{2}-2} \quad (26)$$

By substituting equations (24), (25) and (26) into equation (22), the associated Legendre equation can be expressed by the equation:

$$(1-\nu^2)\frac{d^2u}{d\nu^2}-2\nu\,(m+1)\frac{du}{d\nu}+\{l(l+1)-m(m+1\}\,u=0 \quad (27)$$

Differential equation (27) to obtain:ν

$$(1-\nu^2)\frac{d}{d\nu}u''-2\nu\{(m+1)+1\}\frac{d}{d\nu}u'+\{l(l+1)-(m+1)(m+2)\}u'=0 \quad (28)$$

By supposing (i) you and (ii) equation (28) becomes:$' = u(m+1)=m$

$$(1-\nu^2)\frac{d^2u}{d\nu^2}-2\nu(m+1)\frac{du}{d\nu}+\{l(l+1)-m(m+1\}u=0 \quad (29)$$

At the time, equation (29) has a solution of the form Legendre function, $m=0P_l(\nu)=\frac{1}{2^l l!}\frac{d^l}{d\nu^l}(\nu^2-1)^l$. For the solution is and for $m=1\,P_l'(\nu)=\frac{dP_l(\nu)}{d\nu}m=2$ the solution is $P_l''(\nu)=\frac{d^2P_l(\nu)}{d\nu^2}$. Sefinite in general for all values with can be formulated$m, 0\le m\le l$

$$\frac{d^m}{d\nu^m}P_l(\nu)=P_l^m(\nu)=(1-\nu^2)^{\frac{m}{2}}\frac{d^m}{d\nu^m}P_l(\nu) \quad (30)$$

called the associated Legendre function. $P_l^m(\nu)=\Theta=(1-\nu^2)^{\frac{m}{2}}\frac{d^m}{d\nu^m}P_l(\nu)$

Result and Discussion

Solution of the polar section Schrodinger equation,

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left[\lambda-\frac{m^2}{\sin^2\theta}\right]\Theta=0 \quad (31)$$

Done by suppose $\nu=\cos\theta$ then $\frac{d\nu}{d\theta}=-\sin\theta$ and $\sin^2\theta=1-\nu^2$. So that the polar Schrodinger equation can be expressed in the form of:

$$(1-\nu^2)\frac{d^2\Theta}{d\nu^2}-2\nu\frac{d\Theta}{d\nu}+\left[\lambda-\frac{m^2}{1-\nu^2}\right]\Theta=0 \quad (32)$$

Using the solution of the associated Legendre equation, namely:

(i) $P_l^m(\nu)=(1-\nu^2)^{\frac{m}{2}}\frac{d^m}{d\nu^m}P_l(\nu)$

(ii) $P_l(\nu)=\frac{1}{2^l l!}\frac{d^l}{d\nu^l}(\nu^2-1)^l$

And by using that relationship

$P_l^{-m}(\nu)=P_l^m(\nu)$

Then the solution to the polar Schrodinger equation can be written as:

$$\Theta_{lm}(\theta)=N_{lm}\,P_l^m(\cos\theta) \quad (33)$$

where $N_{lm}=(-1)^{(m+[m])/2}\sqrt{\frac{2l+1}{2}\frac{(l-[m])!}{(l+[m])!}}$ is the normalization constant.

Using the Legendre function and the associated Legendre function, the polar wave function of the Helium ion, ${}^4_2He^+$ in quantum numbers is $(l,m)=(0,0)$

(i) $P_l^m(\nu)=P_0^0(\nu)=\frac{d^0}{d\nu^0}P_0(\nu)$

Because $P_0(\nu)=1$

So $P_0^0(\nu)=P_0(\nu)=1$

(ii) $N_{00}=(-1)^0\sqrt{\frac{1}{2}}=\sqrt{\frac{1}{2}}$

(iii) (iii) $\Theta_{00}(\theta)=\sqrt{\frac{1}{2}}$

For quantum numbers $(l,-1)$, the polar wave function is

(iv) $P_l^m(\nu)=P_0^0(\nu)=\frac{d^0}{d\nu^0}P_0(\nu)$

Because $P_0(\nu)=1$

So $P_0^0(\nu)=P_0(\nu)=1$

(v) $N_{00} = (-1)^0\sqrt{\frac{1}{2}} = \sqrt{\frac{1}{2}}$

(vi) (iii) $\Theta_{00}(\theta) = \sqrt{\frac{1}{2}}$

For quantum numbers $(l, m) = (1, -1)$, the function of the polar wave is

(i) $P_l^m(\nu) = P_1^{-1}(\nu) = \frac{d^1}{d\nu^1}P_1(\nu)$

Because $P_1(\nu) = \cos\theta$

So $P_1^{-1}(\nu) = \sin\theta$

(ii) $N_{00} = (-1)^0\sqrt{\frac{3}{4}}$

(iii) $\Theta_{1(-1)}(\theta) = \frac{\sqrt{3}}{2}\sin\theta$

For quantum numbers $(l, m) = (1,0)$, the polar wave function is

(i) $P_l^m(\nu) = P_1^0(\nu) = \frac{d^0}{d\nu^0}P_1(\nu)$

Because $P_1(\nu) = \cos\theta$

So $P_1^0(\nu) = 1$

(ii) $N_{10} = (-1)^0\sqrt{\frac{3}{2}}$

(iii) $\Theta_{10}(\theta) = \sqrt{\frac{3}{2}}\cos\theta$

For quantum numbers $(l, m) = (1,1)$, the polar wave function is

(i) $P_l^m(\nu) = P_1^1(\nu) = \frac{d^1}{d\nu^1}P_1(\nu)$

Because $P_1(\nu) = \cos\theta$

So $P_1^1(\nu) = \sin\theta$

(ii) $N_{11} = (-1)^1\sqrt{\frac{3}{4}}$

(iii) $\Theta_{11}(\theta) = -\frac{\sqrt{3}}{2}\sin\theta$

For quantum numbers $(l, m) = (2, -2)$, the polar wave function is

(i) $P_l^m(\nu) = P_2^{-2}(\nu) = \frac{d^2}{d\nu^2}P_2(\nu)$

Because $P_2(\nu) = \frac{1}{2}(3\ \cos^2\theta - 1)$

So $P_2^{-2}(\nu) = 3\sin^2\theta$

(ii) $N_{2(-2)} = (-1)^0\sqrt{\frac{5}{48}}$

(iii) $\Theta_{2(-2)}(\theta) = \frac{\sqrt{15}}{4}\sin^2\theta$

For quantum numbers $(l, m) = (2, -1)$, the polar wave function is

(i) $P_l^m(\nu) = P_2^{-1}(\nu) = \frac{d^1}{d\nu^1}P_2(\nu)$

Because $P_2(\nu) = \frac{1}{2}(3\ \cos^2\theta - 1)$

So $P_2^{-1}(\nu) = 3\sin\theta\cos\theta$

(ii) $N_{2(-1)} = (-1)^0\sqrt{\frac{5}{12}}$

(iii) $\Theta_{2(-1)}(\theta) = \frac{\sqrt{15}}{2}\sin\theta\cos\theta$

For quantum numbers $(l, m) = (2, 0)$, the polar wave function is

(i) $P_l^m(\nu) = P_2^0(\nu) = \frac{d^0}{d\nu^0}P_2(\nu)$

Because $P_2(\nu) = \frac{1}{2}(3\ \cos^2\theta - 1)$

So $P_2^0(\nu) = \frac{1}{2}(3\ \cos^2\theta - 1)$

(ii) $N_{20} = (-1)^0\sqrt{\frac{5}{2}}$

(iii) $\Theta_{20}(\theta) = \sqrt{\frac{5}{2}}\frac{1}{2}(3\ \cos^2\theta - 1)$

For quantum numbers $(l, m) = (2,1)$, the polar wave function is

(i) $P_l^m(\nu) = P_2^1(\nu) = \frac{d^1}{d\nu^1}P_2(\nu)$

Because $P_2(\nu) = \frac{1}{2}(3\ \cos^2\theta - 1)$

So $P_2^1(\nu) = 3\sin\theta\cos\theta$

(ii) $N_{21} = (-1)^1\sqrt{\frac{5}{12}}$

(iii) $\Theta_{21}(\theta) = -\frac{\sqrt{15}}{2}\sin\theta\cos\theta$

The polar function of the Helium ion atom expresses the angle formed by the electron position vector relative to the starting point of the coordinate system which is the position of the helium ion nucleus with the z-axis. So it is worth . The polar function and azimuth function are 2 functions related to the value of magnetic quantum numbers and are known as spherical harmonic functions, namely:$({}_2^4He^+)\theta 0\ \le \theta \le\ \pi\Theta(\theta)m$

$$Y_{l,m}(\theta,\phi) = \Theta_{l,m}(\theta)\Phi_m(\phi) \tag{34}$$

with

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi} \tag{35}$$

So that the complete function of the angulatory helium ion race for quantum numbers is shown in table 2.$Y_{l,m}(\theta,\phi) = \Theta_{l,m}(\theta)\Phi_m(\phi) l \le 3$

Table 2. Results of Legendre Polynomial Functions Associated with Polar Helium Ions Using Schrödinger's Equation

l	m	$\Theta_{lm}(\theta)$	$\Phi_m(\phi)$
0	0	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2\pi}}$
	-1	$\frac{\sqrt{3}}{2}\sin\theta$	$\sqrt{\frac{1}{2\pi}}e^{-i\phi}$
1	0	$\sqrt{\frac{3}{2}}\cos\theta$	$\sqrt{\frac{1}{2\pi}}$
	1	$-\frac{\sqrt{3}}{2}\sin\theta$	$\sqrt{\frac{1}{2\pi}}e^{-i\phi}$
2	-2	$\frac{\sqrt{15}}{4}\sin^2\theta$	$\sqrt{\frac{1}{2\pi}}e^{-2i\phi}$

l	m	$\Theta_{lm}(\theta)$	$\Phi_m(\phi)$
	-1	$\frac{\sqrt{15}}{2}\sin\theta\cos\theta$	$\sqrt{\frac{1}{2\pi}}e^{-i\phi}$
	0	$\sqrt{\frac{5}{8}}(3\cos^2\theta-1)$	$\sqrt{\frac{1}{2\pi}}$
	1	$-\frac{\sqrt{15}}{2}\sin\theta\cos\theta$	$\sqrt{\frac{1}{2\pi}}e^{i\phi}$
	2	$\frac{\sqrt{15}}{4}\sin^2\theta$	$\sqrt{\frac{1}{2\pi}}e^{2i\phi}$
	-3	$\sqrt{\frac{7}{1440}}\ 15\ \sin^3\theta$	$\sqrt{\frac{1}{2\pi}}e^{-3i\phi}$
	-2	$\sqrt{\frac{7}{240}}\ 15\ \sin^2\theta\ \cos\theta$	$\sqrt{\frac{1}{2\pi}}e^{-2i\phi}$
	-1	$\sqrt{\frac{7}{24}\frac{3}{2}}\sin\theta\ (5\cos^2\theta-1)$	$\sqrt{\frac{1}{2\pi}}e^{-i\phi}$
	0	$\sqrt{\frac{7}{2}\frac{1}{2}}(5\cos^3\theta-3\ \cos\theta)$	$\sqrt{\frac{1}{2\pi}}$
	1	$-\sqrt{\frac{7}{24}\frac{3}{2}}\sin\theta\ (5\cos^2\theta-1)$	$\sqrt{\frac{1}{2\pi}}e^{i\phi}$
3	2	$\sqrt{\frac{7}{240}}\ 15\ \sin^2\theta\ \cos\theta$	$\sqrt{\frac{1}{2\pi}}e^{2i\phi}$
	3	$-\sqrt{\frac{7}{1440}}\ 15\ \sin^3\theta$	$\sqrt{\frac{1}{2\pi}}e^{3i\phi}$

The angulatory wave function of helium ions on quantum numbers as table 2 is in accordance with the results of research by Supriadi et al. (2019) on hydrogen atoms, Karomah et al. (2021) on Deuterium atoms and Ningrum et al. (2023) on Tritium atoms in momentum space. The angulatory wave function consists of a polar function that shows the orientation of the electron's motion around the helium ion nucleus in the XY plane and an azimuth function that shows rotational motion on the Z-axis. Based on equation (33) it is seen that the polar wave function depends on the value of the quantum number of orbitals, determines the number of sub-shells and the magnetic quantum number. determines the angular momentum of electron orbitals in the direction of the Z-axis. lm

The number of polar wave functions is largely determined by the principal quantum number. At that time and so there is only 1 polar function. For there is a polar wave function. For so there are 9 polar gelombag functions. And for then and. There are 16 polar wave functions, starting from $nn = 1l = 0m = 0n = 2\ (l = 0,1\ ; m = -1,0,1)n = 3\ (l = 0,1,2\ ; m = -2,-1,0,1,2)n = 4l = 0,1,2,3m = -3,-2,-1,0,1,2,3\Theta_{00}(\theta) = \sqrt{\frac{1}{2}}\Theta_{33}(\theta) = -\sqrt{\frac{7}{1440}}\ 15\ \sin^3\theta.$

Under the state, the electron is at a ground energy level with $(n, l, m) = (1,0,0)$ *an s* orbital and no angular component of magnetic momentum. This corresponds to Atkin & Paula (2006: 302) that there is no angular node around the z-axis for a function with ml = 0, which means there is no orbital angular momentum component around that axis. The greatest probability of finding a 1s electron in a helium ion lies along the z-axis and intersects the xy plane. This corresponds to the orbital symmetry characteristic of the sphere as shown by figure 1.

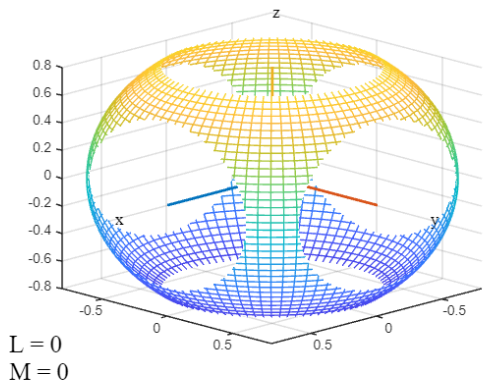

Figure 1. Graph of angular functions on quantum numbers $(\boldsymbol{l},\boldsymbol{m}) = (\mathbf{0},\mathbf{0})$

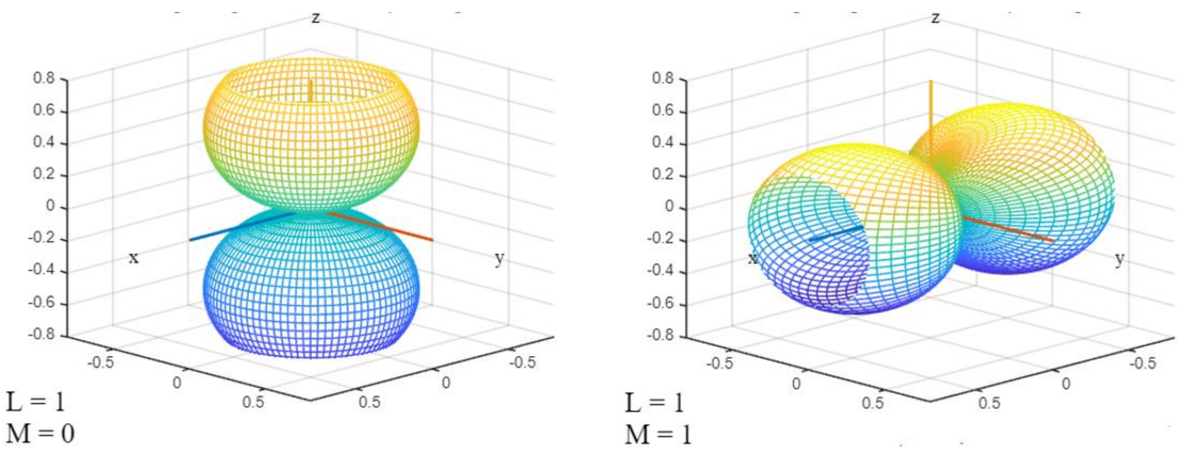

Figure 2. Graph of angular functions on quantum numbers and $(\boldsymbol{l},\boldsymbol{m}) = (\mathbf{1},\mathbf{0})(\boldsymbol{l},\boldsymbol{m}) = (\mathbf{1},\pm\mathbf{1})$

The three-dimensional shape of the$2p_z$ orbital is obtained by rotating the cross section around the z-axis. This results in two distorted ellipsoids, one above and another below the xy plane. The $2p_x$, $2p_y$, and $2p_z$

orbitals have the same shape but different spatial orientations. The two distorted ellipsoids are located on the x-axis for the $2p_x$ orbital, on the y-axis for the $2p_y$ orbital, and on the z-axis for the $2p_z$ orbital (Levine, 2009: 645). Figure 2 illustrates electrons in states $(n, l, m) = (2, 1, 0)$ or in orbitals $2p_z$ that represent the probability of electrons being found along the positive z and negative z axes. While in the state $(n, l, m) = (2, 1, 1)$, electrons have different orbital levels, namely $2p_x$ or $2p_y$. Then electrons are most likely to be found along the x-axis or y-axis.

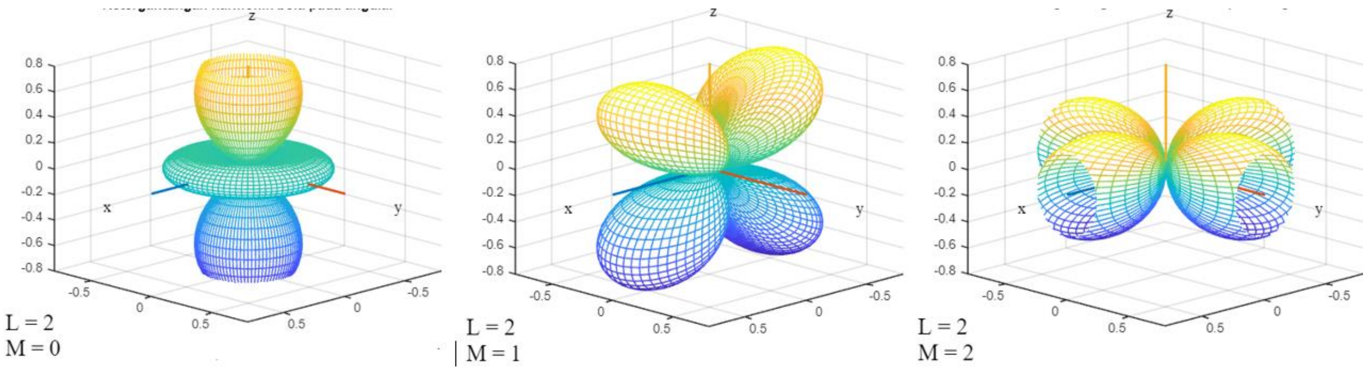

Figure 3. Graph of Angular Functions on quantum numbers and $(\boldsymbol{l}, \boldsymbol{m}) = (\mathbf{2}, \mathbf{0})$; $(\boldsymbol{l}, \boldsymbol{m}) = (\mathbf{2}, \pm\mathbf{1})(\boldsymbol{l}, \boldsymbol{m}) = (\mathbf{2}, \pm\mathbf{2})$

If n equals 3, the possible values of the azimuth quantum number (l) are 0, 1, or 2. As a result, these energy levels include one 3s orbital, three 3p orbitals, and five 3d orbitals. Five d-orbitals have magnetic quantum numbers (ml) +2, +1, 0, −1, −2, representing different angular moments around the z-axis (but of the same magnitude, since in each case l = 2) (Atkin & Paula, 2006: 334). Figure 3 illustrates electrons at the orbital state $(n, l, m) = (3, 2, 0)$ or $3d$ level, showing the probability density distribution with a maximum along the z-axis. For state $(n, l, m) = (3, 2, \pm 1)$ indicates the probability with a maximum value along a given direction in the space corresponding to the x and y axes. As for the state $(n, l, m) = (3, 2, \pm 2)$ shows the greatest probability is found in certain regions along the positive z-axis or along the negative z-axis.

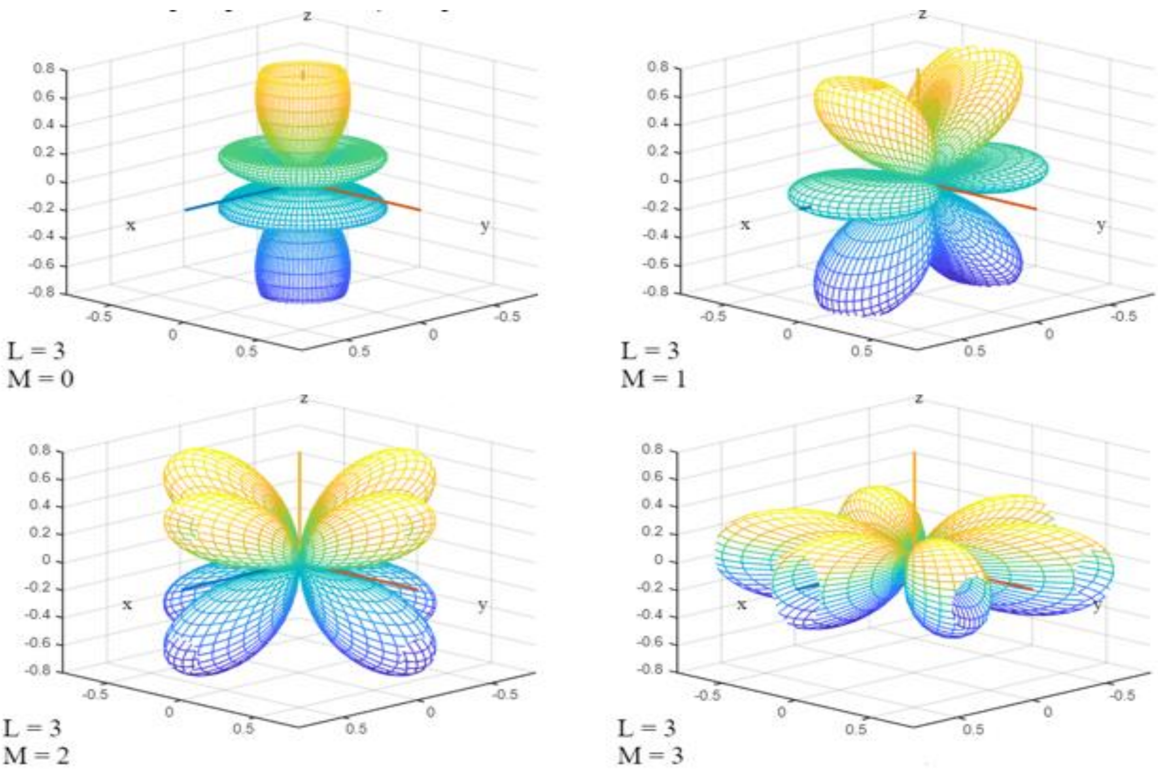

Figure 4. 3D angular graph for the sets of quantum number (n=4) with (l=3,m=0), (l=3,m=±1), (l=3,m=±2), and (l=3,m=±3)

In the state $(n, l, m) = (4, 3, 0)$, the electron is at the level of the 4f orbital. From the graph it can be seen that the maximum probability of finding electrons is along the z-axis. For the $(n, l, m) = (4, 3, \pm 1)$ situation shows 3 different orientations of the 4f orbital where electrons are most likely to be found. And also from the chart it is seen that the maximum probability is along the direction of the x or y axis. The $(n, l, m) = (4, 3, \pm 2)$ situation shows that there are two orientations of these orbitals. The greatest chance of finding electrons along the positive z-axis $(m = +2)$ or along the negative z-axis $m = -2$. And under the state $(n, l, m) = (4, 3, \pm 3)$, the greatest probability is found along the positive z-axis or negative z-axis.

Conclusion

Based on the results of research that has been done, the anguler wave function of Helium ions $({}_{2}^{4}He^{+})$ in momentum space is the same as the wave function of Hydrogen atoms and deuterium atoms in position space and is also in the same shape as the anguler wave function of Tritium atoms in momentum space. So the angulatory wave function of a hydrogenic atom and has the same angulatory function both in position space and in momentum space.

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Author Contributions

In this research, all authors contributed according to their respective assignments. BS, EDK, RW, and MF contributed to designing research, conducting research, collecting data, analyzing data, and writing articles. Then MNH, ECR, and HA contributed to correcting results, inputting data, and contributing to completing the reports that had been prepared.

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Conflicts of Interest

The author declares that there were no conflicts of interest regarding the publication of this article.

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