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The Existence of Cocyclic Formed by Angle Trisectors in a Butterfly Quadrilateral and its Application in Physics

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Abstract: One of the most intriguing results related to angle trisectors in convex quadrilaterals is Morley's Theorem, which several authors have subsequently extended to non-convex quadrilaterals. Numerous studies have explored the side lengths of angle bisectors, angle trisectors, and the area ratios formed by angle trisectors in both convex and non-convex quadrilaterals. However, no research has discussed the problem of angle trisectors in butterfly quadrilaterals. Therefore, this paper aims to extend the concept of angle trisectors to butterfly quadrilaterals. Various other quadrilaterals will be formed from the construction of these angle trisectors. By employing the concept of concyclic, we will demonstrate the existence of several concyclic quadrilaterals arising from this trisector construction. Triangle angles in butterfly quadrilaterals in geometry play an important role in physics. Geometry provides a visual and mathematical language that allows us to describe, analyze, and understand various physical phenomena.

Keywords: Angle trisector; Butterfly quadrilateral; Cyclic quadrilateral

Introduction

In a planar geometric shape, various types of bisectors can be identified, including angular divider lines. These angular divider lines are classified into two types: the angle bisector and the angle trisector. An angle bisector is a line that divides an angle into two equal parts. An angle trisector, on the other hand, consists of two lines that divide an angle into three equal parts. The discussion on angle trisectors has been examined in several papers (Florio, 2023). In general, the discourse on angle trisectors is categorized into two types: angle trisectors in triangles and angle trisectors in quadrilaterals. One of the discussions on angle trisectors in triangles is presented in (Dergiades & Hung, 2020). This paper discusses methods for determining the length of the angle trisector line in a triangle and the area ratios of triangles formed by the angle trisectors. To determine the length of an angle trisector in a triangle, an altitude line is utilized (Jumianti et al., 2021). This altitude line is then combined with trigonometry, allowing the length of the angle trisector to be determined based on the area of the original triangle. Furthermore, for the area ratio, the sine rule is applied. Additionally, discusses a theorem that applies angle trisectors to the three arbitrary angles of a triangle, namely Morley's Theorem.

In Morley's Theorem, inner angle trisectors of the triangle are used, where their intersections form an equilateral triangle. However, in this study, outer angle trisectors and extended angles of the triangle are employed, resulting in intersections that also form an equilateral triangle. Subsequently, explores further development of Morley's Theorem from (Brailas, 2024; Büchi, 2024; Trinh et al., 2024; Volpe et al., 2023). In that article, the angle trisectors are combined with cyclicity to produce an equilateral triangle. Then, Blåsjö (2022) and Carrignon et al. (2020), discusses a development of Morley's Theorem, namely the use of angle trisectors in right triangles. This paper explains that, to obtain an equilateral triangle, inner and outer angle trisectors are used on the non-right angles. What distinguishes this work from other discussions is the use of supplementary angle trisectors on non-right angles, resulting in a parallelogram. The discussion of angle trisectors in

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quadrilaterals is found in (Csiba et al., 2024). Article Barton (2024) examines angle trisectors applied to specific types of quadrilaterals, where intersections of the angle trisectors also form special quadrilaterals. For example, in the case of a rectangle, angle trisectors at all four vertices intersect to form a rhombus. Another paper discussing angle trisectors in quadrilaterals is presented in (Kegel & Schmäschke, 2024).

This study discusses methods for determining the length of the angle trisectors and the area ratios of triangles formed in both convex and non-convex quadrilaterals. To determine the length of the angle trisectors and their area ratios, the sine rule is used. In determining the length of the angle trisectors, several conditions are considered (Gu et al., 2021; Chen et al., 2020): both angle trisectors intersecting on the same side of the quadrilateral, both trisectors intersecting on two different sides of the quadrilateral, and one trisector passing through one of the quadrilateral's original vertices. In this paper, butterfly quadrilaterals will be discussed, with angle trisectors applied at each of their vertices. Since two of the four angles of the butterfly quadrilateral measure more than 180°, the angle trisectors will be applied to the complementary angles of these two vertices. Subsequently, these angle trisectors will intersect the sides of the butterfly quadrilateral. From the points of intersection, four distinct cyclic quadrilaterals are formed.

Method

Before delving into the core discussion, there are several topics that need to be understood. These topics include the definitions and theorems related to angle trisectors, the sine rule (the method that will be employed in the proofs), and the theorem concerning cyclic quadrilaterals.

Angle Trisector

The discourse presented in this paper is fundamentally linked to the concept of the angle trisector. An angle trisector is a type of line that divides an angle in a planar figure. In Son (2023), the definition of an angle trisector is stated as follows:

Definition 2.1.1. Angle trisector has two dividing lines that divide the angle into three equal parts. In Figure 1, it can be observed that in any ΔABC with $BC = a, AC = b, AB = c, \angle BAC = \alpha, \angle ABC = \beta,$ and $\angle BCA = \gamma$, there are lines AA_1 and AA_2 drawn from $\angle BAC$. The lines AA_1 and AA_2 are referred to as angle trisectors if $\angle BAA_1 = \angle A_1AA_2 = \angle A_2AC = \frac{3}{\alpha}$. Furthermore, in a theorem is discussed to determine the length of the trisector line in a triangle. The area of the triangle is utilized to ascertain the length of its angle trisector. The theorem is as follows:

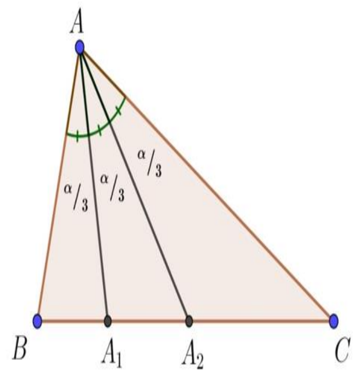

Figure 1. The angle trisectors at $\angle BAC$ in ΔABC

Theorem 2.1.1. In ΔABC, the angle trisectors AA_1 and AA_2 that divide angle A with $BC = a$, $AC = b$, $AB = c$, and $\angle A = \alpha$, $\angle B = \beta$, and $\angle C = \gamma$, have lengths given by:

$$AA_1 = \frac{2L}{a \sin\left(\frac{\alpha}{3}+\beta\right)}, \quad (1)$$

and

$$AA_2 = \frac{2L}{a \sin\left(\frac{\alpha}{3}+\gamma\right)}. \quad (2)$$

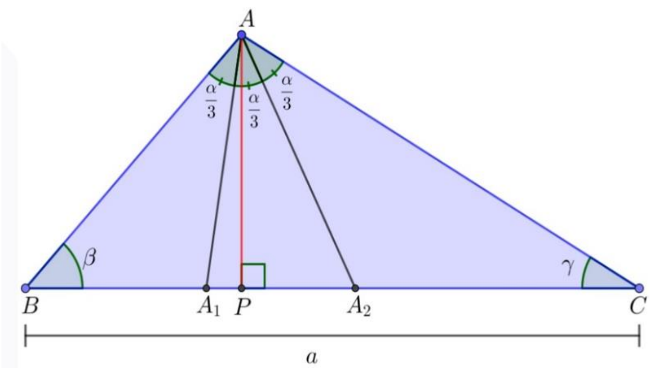

Figure 2. The angle trisectors and altitudes in ΔABC proof. see

Sine Rule

The sine rule is one of the fundamental theorems in trigonometry that illustrates the relationship between the lengths of the sides of a triangle and the sine functions of the corresponding angles. In the context of any triangle, the sine rule involves three ratios that connect the lengths of the sides with the opposite angles. When information regarding the angles and lengths of sides is provided, the length of the remaining side can be determined using the sine rule. For instance, if the lengths of two sides and one angle of a triangle are known, the sine rule facilitates the calculation of the length of the third side. Furthermore, the sine rule is also associated with the circumradius of the triangle, which demonstrates the relationship between the lengths of the sides and the circumradius. By comprehending the sine rule, geometric problems involving triangles can be effectively solved. Mathematically, the sine rule is presented in (Guo, 2022). Figure Theorem 2.2.1. Let a, b, and c be the lengths of the sides inΔABC, then the following holds:

$$\frac{a}{\sin\angle A}=\frac{b}{\sin\angle B}=\frac{c}{\sin\angle C}=2R \quad (3)$$

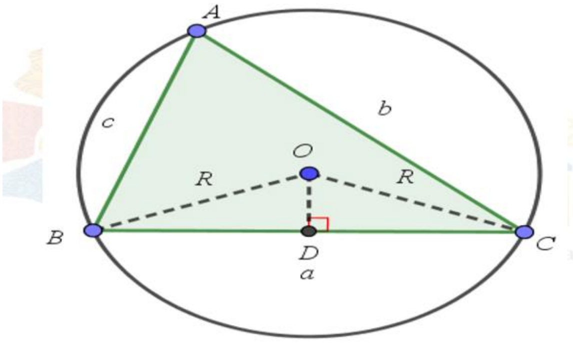

Figure 3. ΔABC and circumcircle of the triangle proof. see

Cyclic Quadrilateral

One of the discussions to be addressed in this paper is the cyclic quadrilateral, or the quadrilateral formed by chords of a circle. As the name suggests, this quadrilateral has sides that are formed by the chords of a single circle. In, the definition of a cyclic quadrilateral is stated as follows (Uygan & Bozkurt, 2021; Unger, 2023): Definition 2.3.1. A cyclic quadrilateral is a quadrilateral whose four vertices lie on a single circle. Definition 2.3.1 is illustrated in Figure 5. In the figure, it can be seen that line segment AB is a chord of a circle. This chord spans points A and B, each of which lies on the circumference of the circle. The same applies to line segments BC, CD, and DA, which are also chords of the same circle. Since all the chords are interconnected, they form a cyclic quadrilateral. A theorem discussing cyclic quadrilaterals is presented. The theorem is as follows: Theorem 2.3.1. Let $ABCD$ be a convex quadrilateral. The following statements are equivalent: $ABCD$ is a cyclic quadrilateral; $\angle BAC=\angle BDC$; $\angle A+\angle C=180°$; $\angle ABE=\angle D$.

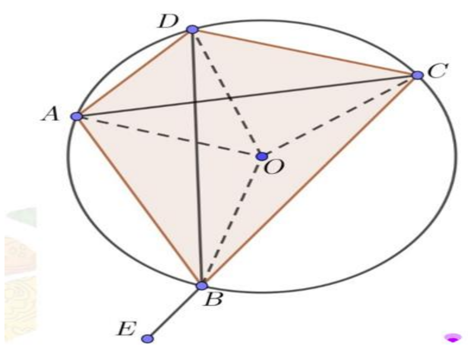

Figure 4. A cyclic quadrilateral proof. see

Result and Discussion

Let us consider a butterfly quadrilateral ABCD with $AB=a$, $BC=b$, $CD=c$, $AD=d$, and $\angle DAB=\alpha$, $\angle ABC=\beta$, $\angle BCD=\gamma^*$, also $\angle CDA=\delta^*$. In this quadrilateral, AA_1 and AA_2 are constructed as angle trisectors at $\angle A$, BB_1 and BB_2 are constructed as angle trisectors at $\angle B$, CC_1 and CC_2 are constructed as angle trisectors for the complement angle at $\angle C$, and DD_1 and DD_2 are constructed as angle trisectors for the complement angle at $\angle D$. From the angle trisectors at the four angles of this butterfly quadrilateral, four cyclic quadrilaterals are obtained as described in the following theorem. Theorem 3.2.1. In the butterfly quadrilateral $ABCD$ with $AB=a$, $BC=b$, $CD=c$, $AD=d$, and $\angle DAB=\alpha$, $\angle ABC=\beta$, $\angle BCD=\gamma^*$, also $\angle CDA=\delta^*$, given the angle trisectors at angles $\alpha,\beta,\gamma=360°-\gamma^*$, and $\delta=360°-\delta^*$, four cyclic quadrilaterals can be formed, namely $A_1B_1D_1C_1$, $A_1B_1D_2C_2$, $A_2B_2D_1C_1$, and $A_2B_2D_2C_2$. P**roof.** Theorem 3.2.1 is illustrated in Figure 5.

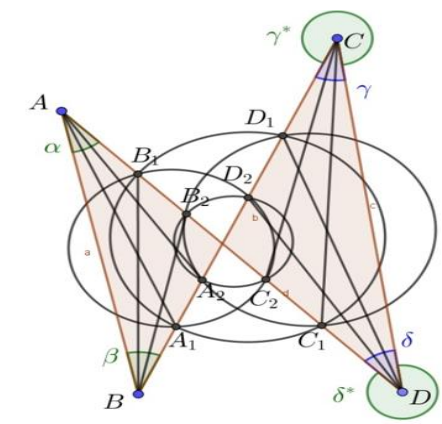

Figure 5. The cyclic quadrilateral on the angle trisectors of a butterfly quadrilateral

Cyclic Quadrilateral $A_1B_1D_1C_1$

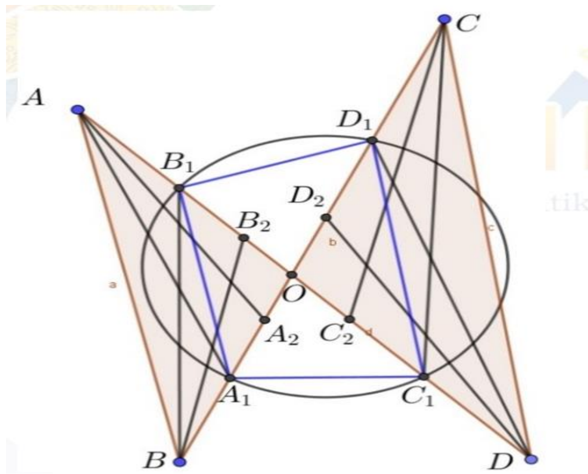

Figure 6. Cyclic quadrilateral $A_1B_1D_1C_1$.

In ΔCD_1D we obtain $\angle CD_1D = 180° - \left(\frac{\delta}{3} + \gamma\right)$. Since $\angle A_1D_1D$ and $\angle CD_1D$ are supplementary, it follows that

$$\angle A_1D_1D + \angle CD_1D = 180° \quad (4)$$
$$\angle A_1D_1D = 180° - \angle CD_1D$$
$$\angle A_1D_1D = \frac{\delta}{3} + \gamma \quad (5)$$

Next, by letting $\angle DD_1C_1 = x$, it is obtained that
$$\angle C_1D_1A_1 + \angle DD_1C_1 = \angle A_1D_1D$$
$$\angle C_1D_1A_1 = \angle A_1D_1D - \angle DD_1C_1$$
$$\angle C_1D_1A_1 = \frac{\delta}{3} + \gamma - x. \quad (6)$$

Then, in ΔCOD, since $\angle OCD = \gamma$ and $\angle ODC = \delta$ it is obtained that
$$\angle COD = 180° - (\gamma + \delta). \quad (7)$$

From equation (2) and equation (3) in ΔOC_1D_1 it is optained that
$$\angle COD + \angle C_1D_1A_1 + \angle B_1C_1D_1 = 180°$$
$$\angle B_1C_1D_1 = \frac{2\delta}{3} + x. \quad (8)$$

Next, $\angle B_1C_1D_1$ and $\angle B_1A_1D_1$ subtend the same chord, B_1D_1, so we obtain $\angle B_1A_1D_1 = \frac{2\delta}{3} + x$. Then, by letting $\angle D_1B_1C_1 = y$, since $\angle D_1B_1C_1$ and $\angle D_1A_1C_1$ subtend the same chord, it is obtained that $\angle D_1B_1C_1 = \angle D_1A_1C_1 = y$. Next, in ΔOB_1D_1 it is obtained that
$$\angle B_1OD_1 + \angle OD_1B_1 + \angle OB_1D_1 = 180°$$
$$\gamma + \delta + \angle OD_1B_1 + y = 180°$$
$$\angle OD_1B_1 = \angle A_1D_1B_1 = 180° - (\gamma + \delta + y). \quad (9)$$

From the obtained angles, it follows that
$$\angle B_1D_1C_1 + \angle B_1A_1C_1 = \angle B_1D_1A_1 + \angle C_1D_1A_1 + \angle B_1A_1D_1 + \angle D_1A_1C_1$$
$$\angle B_1D_1C_1 + \angle B_1A_1C_1 = 180° - (\gamma + \delta + y) + \gamma + \frac{\delta}{3} - x + \frac{2\delta}{3} + x + y$$
$$\angle B_1D_1C_1 + \angle B_1A_1C_1 = 180°. \quad (10)$$

Thus, Theorem 3.2.1 for the cyclic quadrilateral $A_1B_1D_1C_1$ is prooven. □

Cyclic Quadrilateral $A_1B_1D_2C_2$

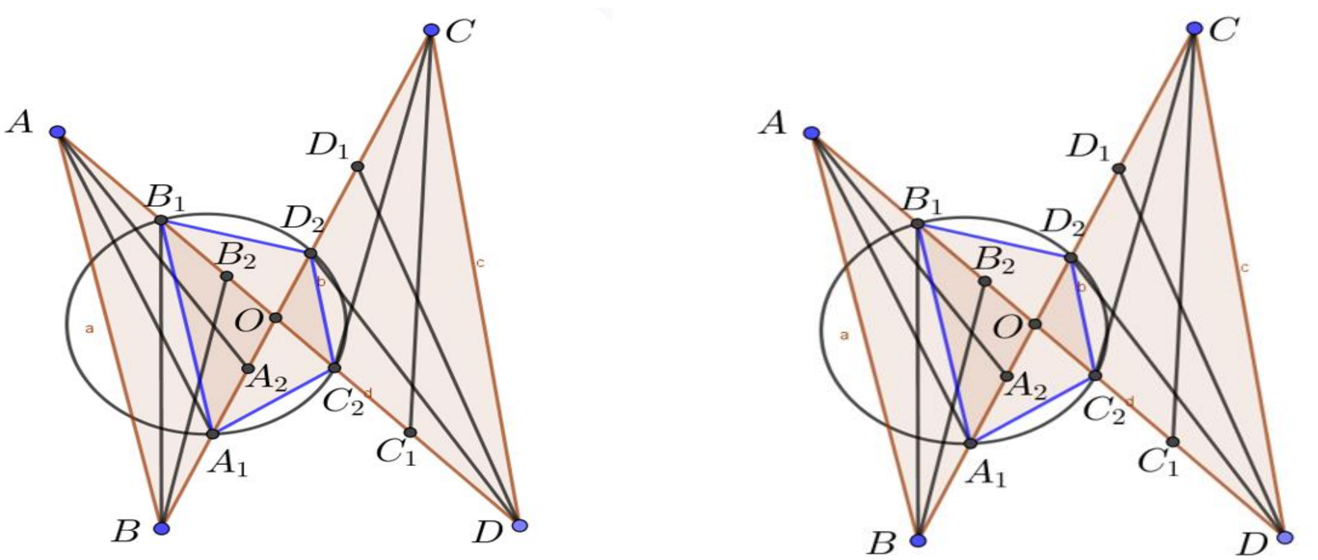

Figure 7. Cyclic quadrilateral $A_1B_1D_2C_2$

In ΔCD_2D we obtain $\angle CD_2D = 180° - \left(\frac{2\delta}{3} + \gamma\right)$. Since $\angle A_1D_2D$ and $\angle CD_2D$ are supplementary, it follows that
$$\angle A_1D_2D + \angle CD_2D = 180°$$
$$\angle A_1D_2D = 180° - \angle CD_2D$$
$$\angle A_1D_2D = \frac{\delta}{3} + \gamma \quad (11)$$

Next, by letting $\angle DD_2C_2 = x$, it is obtained that
$$\angle C_2D_2A_1 + \angle DD_2C_2 = \angle A_1D_2D$$
$$\angle C_2D_2A_1 = \angle A_1D_2D - \angle DD_2C_2$$
$$\angle C_2D_2A_1 = \frac{2\delta}{3} + \gamma - x. \quad (12)$$

Then, in ΔCOD, since $\angle OCD = \gamma$ and $\angle ODC = \delta$ it is obtained that
$$\angle COD = 180° - (\gamma + \delta). \quad (9)$$

From equation (8) and equation (9) in ΔOC_2D_2 it is obtained that
$$\angle COD + \angle C_2D_2A_1 + \angle B_1C_2D_2 = 180°$$
$$\angle B_1C_2D_2 = \frac{\delta}{3} + x. \quad (13)$$

Next, $\angle B_1C_2D_2$ and $\angle B_1A_1D_2$ subtend the same chord, B_1D_2, so we obtain $\angle B_1A_1D_2 = \frac{\delta}{3} + x$. Then, by letting $\angle D_2B_1C_2 = y$, since $\angle D_2B_1C_2$ and $\angle D_2A_1C_2$ subtend the same chord, it is obtained that $\angle D_2B_1C_2 = \angle D_2A_1C_2 = y$. Next, in ΔOB_1D_2 it is obtained that
$$\angle B_1OD_2 + \angle OD_2B_1 + \angle OB_1D_2 = 180°$$
$$\gamma + \delta + \angle OD_2B_1 + y = 180°$$
$$\angle OD_2B_1 = \angle A_1D_2B_1 = 180° - (\gamma + \delta + y). \quad 14)$$

From the obtained angles, it follows that

$\angle B_1D_2C_2 + \angle B_1A_1C_2$
$= \angle B_1D_2A_1 + \angle C_2D_2A_1 + \angle B_1A_1D_2 + \angle D_2A_1C_2$
$\angle B_1D_2C_2 + \angle B_1A_1C_2$
$= 180° - (\gamma + \delta + y) + \gamma + \frac{2\delta}{3} - x + \frac{\delta}{3} + x + y$
$\angle B_1D_2C_2 + \angle B_1A_1C_2 = 180°.$ (15)
Thus, Theorem 3.2.1 for the cyclic quadrilateral $A_1B_1D_2C_2$ is proven. □

Cyclic Quadrilateral $A_2B_2D_1C_1$.

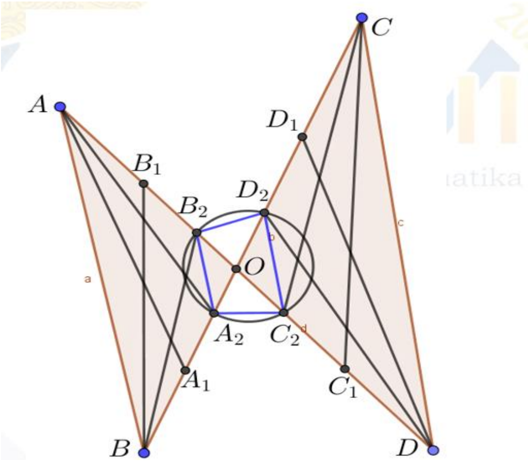

Figure 8. Cyclic quadrilateral $A_2B_2D_2C_2$

In ΔCD_2D we obtain $\angle CD_2D = 180° - \left(\frac{2\delta}{3} + \gamma\right)$. Since $\angle A_2D_2D$ and $\angle CD_2D$ are supplementary, it follows that
$\angle A_2D_2D + \angle CD_2D = 180°$
$\angle A_2D_2D = 180° - \angle CD_2D$

$\angle A_2D_2D = \frac{2\delta}{3} + \gamma$ (16)
Next, by letting $\angle DD_2C_2 = x$, it is obtained that
$\angle C_2D_2A_2 + \angle DD_2C_2 = \angle A_2D_2D$
$\angle C_2D_2A_2 = \angle A_2D_2D - \angle DD_2C_2$
$\angle C_2D_2A_2 = \frac{2\delta}{3} + \gamma - x$ (17)
Then, in ΔCOD, since $\angle OCD = \gamma$ and $\angle ODC = \delta$ it is obtained that
$\angle COD = 180° - (\gamma + \delta).$ (18)
From equation (20) and equation (21) in ΔOC_2D_2 it is obtained that
$\angle COD + \angle C_2D_2A_2 + \angle B_2C_2D_2 = 180°$
$\angle B_2C_2D_2 = \frac{\delta}{3} + x.$ (19)

Next, $\angle B_2C_2D_2$ and $\angle B_2A_2D_2$ subtend the same chord, B_2D_2, so we obtain $\angle B_2A_2D_2 = \frac{\delta}{3} + x$. Then, by letting $\angle D_2B_2C_2 = y$, since $\angle D_2B_2C_2$ and $\angle D_2A_2C_2$ subtend the same chord, it is obtained that $\angle D_2B_2C_2 = \angle D_2A_2C_2 = y$. Next, in ΔOB_2D_2 it is obtained that
$\angle B_2OD_2 + \angle OD_2B_2 + \angle OB_2D_2 = 180°$
$\gamma + \delta + \angle OD_1B_2 + y = 180°$
$\angle OD_2B_2 = \angle A_2D_2B_2 = 180° - (\gamma + \delta + y).$ (20)
From the obtained angles, it follows that
$\angle B_2D_2C_2 + \angle B_2A_2C_2$
$= \angle B_2D_2A_2 + \angle C_2D_2A_2 + \angle B_2A_2D_2 + \angle D_2A_2C_2$
$\angle B_2D_2C_2 + \angle B_2A_2C_2$
$= 180° - (\gamma + \delta + y) + \gamma + \frac{2\delta}{3} - x + \frac{\delta}{3} + x + y$
$\angle B_2D_2C_2 + \angle B_2A_2C_2 = 180°.$ (21)
Thus, Theorem 3.2.1 for the cyclic quadrilateral $A_2B_2D_2C_2$ is proven.

Constructing trisectors within a triangle efficiently illustrates the notion of concyclic quadrilaterals, resulting in diverse configurations that exhibit concyclic characteristics (Ida et al., 2015). This construction demonstrates the formation of several quadrilaterals, each displaying cyclic properties. Construction of trisections and cyclic quadrilaterals (Lupenko, 2024). The intersection points of the trisectors of a triangle can function as vertices for various quadrilaterals (Andrica & Bagdasar, 2024). These points originate from the triangle's angles, resulting in distinct configurations that preserve cyclic features (Shen et al., 2023; Ma et al., 2021; Zhao et al., 2023). Cyclic Properties: By confirming that opposite angles are supplementary, a necessary condition for cyclicity, we can show that the quadrilaterals formed from these intersections are cyclic (Golewski, 2023; Lu et al., 2023; Morseletto, 2020).

Diagonal relationships three diagonal configurations: For any cyclic quadrilateral, three unique configurations arise from the arrangement of its sides. Different arrangements create different diagonal lengths, which shows that the quadrilaterals made by the intersections of the trisectors are cyclical (Vízek et al., 2023). Geometric Inequalities: By articulating the connections between the sides and diagonals of these quadrilaterals, geometric inequalities reinforce their cyclic properties (Cybulski et al., 2024). Triangle angles in butterfly quadrilaterals in geometry play an important role in physics. Geometry provides a visual and mathematical language that allows us to describe, analyze, and understand various physical phenomena. Here are some examples of applications of geometry in physics:

Mechanics

Kinematics: Studying the motion of objects without considering the cause. Concepts such as distance, displacement, velocity, and acceleration are visualized and calculated using geometry.

Dynamics

Studying the relationship between force and motion. Force diagrams, which are visual representations of the forces acting on an object, use geometric concepts to determine the resultant force (Leite et al., 2021; Lamanepa et al., 2022).

Optics

Optical geometry: Studying the properties of light such as reflection and refraction. Concepts such as angle of incidence, angle of reflection, and the formation of images by mirrors and lenses use geometric principles.

Electromagnetism

Electric and magnetic fields: Electric and magnetic field lines that describe the direction and strength of the field, are described using geometric concepts.

Electric potential

The concept of electric potential related to electric potential energy is visualized using equipotential surfaces which are geometric surfaces.

Quantum mechanics

Wave function: The wave function that describes the state of a particle in quantum mechanics, often visualized in three-dimensional space (Figueiras et al., 2019).

Relativity

Space-time geometry: Einstein's general theory of relativity uses non-Euclidean geometry to describe gravity as the curvature of space-time. In addition, geometry is also used in (Cabral et al., 2020).

Particle physics

To describe the interactions between subatomic particles. Cosmology (Addazi et al., 2022; Moghaddasi & Yousefnia, 2024; Kousar, 2020): To study the structure and evolution of the universe; Condensed matter physics: To study the properties of matter at the atomic scale. In general, geometry provides a powerful framework for: Visualizing physical phenomena: Making physical models easier to understand (Kim, 2019; Oughton et al., 2024). Analyzing physical problems: Using mathematical tools based on geometry to solve equations and find solutions; Formulating physical theories: Developing new theories that are consistent with experimental data. So, geometry is not just a branch of mathematics, but also a very important tool in understanding the physical world around us (Kupczynski, 2024; Tong et al., 2024).

Conclusion

From the elaboration above, two conditions can be drawn. First, in the butterfly quadrilateral $ABCD$ with $AB = a$, $BC = b$, $CD = c$, $AD = d$, and $\angle DAB = \alpha$, $\angle ABC = \beta$, $\angle BCD = \gamma^*$, also $\angle CDA = \delta^*$, when the angle trisectors are drawn for angles α, β, $\gamma = 360° - \gamma^*$, and $\delta = 360° - \delta^*$, the intersections of the angle trisectors can form four cyclic quadrilaterals: $A_1B_1D_1C_1$, $A_1B_1D_2C_2$, $A_2B_2D_1C_1$, and $A_2B_2D_2C_2$.

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Author Contributions

Conceptualization; R. A.; methodology.; I. W.; validation; R. D. S.; formal analysis; U.; investigation.; R. A.; resources; R. A.; data curation: I. W.; writing—original draft preparation. R. D. S.; writing—review and editing: U.; visualization: R. A. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest

The authors declare no conflict of interest.

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